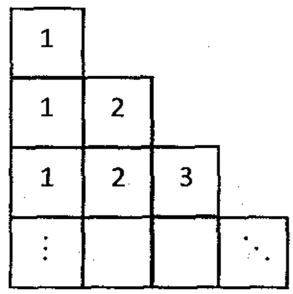
2017 NYJC Prelim Paper 1

1 A board is such that the n^{th} row from the top has n tiles, and each row is labelled from left to right in ascending order such that the i^{th} tile is labelled i, where n and i are positive integers.



Given that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, by finding the sum of the numbers in the r^{th} row, show that the sum of all the numbers in n rows of tiles is $\frac{1}{6}(n)(n+1)(n+2)$. [4]

2 The curve C has equation $2x - y^2 = (x+y)^2$.

(i) Find the equations of the tangents to C which are parallel to the x-axis. [4]

(ii) The line l is tangent to C at $A(2,-2)$. If the normal to C at the origin O meets l at the point B, find the area of triangle OAB. [4]

3 **Do not use a calculator in answering this question.**

(i) Explain why the equation $z^3 + az^2 + az + 7 = 0$ cannot have more than two non-real roots, where a is a real constant. [1]

(ii) Given that $z = -7$ is a root of the equation in **(i)**, find the other roots, leaving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

(iii) Hence, solve the equation $iz^3 + 8z^2 - 8iz - 7 = 0$, leaving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

4 (i) By using the substitution $x-1=3\tan\theta$, find $\displaystyle\int \frac{1}{\sqrt{x^2-2x+10}}\,dx$. [5]

(ii) By expressing $x+3=A(2x-2)+B$, find $\displaystyle\int \frac{x+3}{\sqrt{x^2-2x+10}}\,dx$. [3]

5 (i) By considering $f(r)-f(r+1)$, where $f(r)=\dfrac{\sqrt{r}}{2\sqrt{r}+1}$, find

$$\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$$

in terms of n. [3]

(ii) Hence, find $\displaystyle\sum_{r=1}^{\infty}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$. [2]

(iii) Find the smallest integer n such that

$$\sum_{r=1}^{n}\frac{\sqrt{r+1}-\sqrt{r+2}}{\left(2\sqrt{r+1}+1\right)\left(2\sqrt{r+2}+1\right)}<-0.1.$$ [3]

6 The curve C has equation

$$y=1+\frac{2x+p}{(x-2)(x+3)},$$

where p is a constant.

(i) Find the range of values of p for which C has more than one stationary point. [4]

(ii) Sketch C for p = 7, stating the coordinates of the turning point(s) and the points of intersection with the axes and the equations of any asymptotes. [3]

(iii) By sketching a suitable graph on the same diagram, solve the inequality

$$1+\sqrt{12-x^2}\geq\frac{2x+7}{(2-x)(x+3)}.$$ [3]

7 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \mathrm{e}^{-x^2}, \quad x \in \mathbb{R},\ x < 0,$$

$$\mathrm{g}: x \mapsto \frac{1}{x+3}, \quad x \in \mathbb{R},\ x \neq -3.$$

(i) Show that g^{-1} exists, and define g^{-1} in a similar form. [2]

(ii) State the solution set for $\mathrm{g}\mathrm{g}^{-1}(x) = x$. [1]

(iii) Explain why fg^{-1} does not exist. [1]

Let the function h be defined by

$$\mathrm{h}: x \mapsto \mathrm{g}(x), \quad x \in \mathbb{R},\ x < k,$$

where k is a real constant.

(iv) Given that $\mathrm{f}\,\mathrm{h}^{-1}$ exists, state the maximum value of k. [1]

(v) For the value of k found in **(iv)**,

(a) find the exact range of $\mathrm{f}\,\mathrm{h}^{-1}$, [2]

(b) solve $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$. [2]

8 A curve C has parametric equations

$$x = 1 + \mathrm{e}^{t} + \mathrm{e}^{-t}, \quad 2y = \mathrm{e}^{t} - \mathrm{e}^{-t}, \quad t \in \mathbb{R}.$$

(i) Show that the Cartesian equation of C is $\dfrac{(x-1)^2}{2^2} - y^2 = 1$. [2]

(ii) Sketch C, showing clearly the equations of any asymptotes and coordinates of the centre and the point(s) where the curve cuts the x-axis. [3]

(iii) Find the exact area of the region bounded by C and the line $x = 1 + \mathrm{e} + \mathrm{e}^{-1}$. [4]

(iv) Find the volume of the solid of revolution when the region bounded by C and the lines $x = 3$ and $y = 4$ is rotated completely about the y-axis. [2]

9 With reference to an origin O, a particle P moves in space with position vector $(\lambda-\mu)\mathbf{i}+(1+2\mu)\mathbf{j}+(2-3\lambda)\mathbf{k}$. Another particle Q moves along the line l with equation $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$, $t\in\mathbb{R}$.

(i) State the locus of P. [1]

(ii) Determine if the particles P and Q can meet. [3]

(iii) Find the shortest possible distance between P and Q. [2]

Another particle R moves along the line m with equation $\mathbf{r}=\begin{pmatrix}1\\k\\0\end{pmatrix}+s\begin{pmatrix}2\\-2\\-3k\end{pmatrix}$, $s\in\mathbb{R}$, where k is a constant.

(iv) Find condition(s) satisfied by k if lines l and m are skew lines. [3]

(v) A particle is shot from $X(0,-1,-5)$ perpendicularly toward the path of Q. Find the coordinates of the point where it crosses the path of Q. [2]

10 A car is travelling at a speed of 30 m/s on a road heading towards a perpendicular train track, which is elevated 30 m above the ground. The front of the car is 40 m away from the track when the front of the train first crossed the road.

If the train is travelling at 20 m/s, show that the distance between the front of the train and the car is $\sqrt{1300t^2-2400t+2500}$ m. [2]

(i) How fast is the front of both the train and the car separating 1 second later? [2]

(ii) Find the distance when the front of the train and the front of the car are closest. [4]

(iii) Find the rate of change of the angle of elevation of the front of the train from the car 1 second later. [4]

11 Suppose a point P on the rim of a wheel of radius r is initially at the point O. As the wheel roll along the x-axis without slippage, the locus of P, known as a *cycloid*, has parametric equations given by

$$x=r(\theta-\sin\theta),\ y=r(1-\cos\theta),\ \theta\geq 0.$$

(i) Sketch the locus of P for $0\leq\theta\leq 4\pi$. [2]

(ii) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\cot\dfrac{\theta}{2}$. [3]

(iii) Show that the curve is a solution to the differential equation $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\dfrac{2r}{y}-1$. [3]

(iv) Find the exact area bounded by the locus of P and the x-axis for $0\leq x\leq 2\pi r$. [4]

———**END OF PAPER**———

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<table>
<tr><th>Qn</th><th></th><th>Remarks</th></tr>
<tr><td>1</td><td>

Sum of numbers in kth row = $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$

$$\begin{aligned}\text{Required sum} &= \sum_{k=1}^{n} \frac{k\left(k+1\right)}{2}\\ &= \frac{1}{2}\sum_{k=1}^{n}\left(k^2+k\right)\\ &= \frac{1}{12}n\left(n+1\right)\left(2n+1\right)+\frac{1}{4}n\left(n+1\right)\\ &= \frac{1}{12}n\left(n+1\right)\left(2n+1+3\right)\\ &= \frac{1}{6}n\left(n+1\right)\left(n+2\right)\end{aligned}$$

</td><td>

A handful of students wrote $\sum_{r=1}^{k} k = \frac{1}{2}k(k+1)$, without realising that k is in fact a constant, and the expression is incorrect.

</td></tr>
<tr><td>2(i)</td><td>

Differentiating $2x - y^2 = \left(x+y\right)^2$ _____(1)
implicitly with respect to x,

$$2-2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(x+y\right)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

Where tangent is parallel to the x-axis, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$2 = 2\left(x+y\right)$$

$$y = 1-x \qquad \text{\_\_\_\_\_(2)}$$

Sub (2) in (1),

$$2x-\left(1-x\right)^2 = \left(x+1-x\right)^2$$

$$x^2-4x+2=0$$

$$x = \frac{-\left(-4\right)\pm\sqrt{\left(-4\right)^2-4\left(2\right)}}{2} = 2\pm\sqrt{2}$$

When $x = 2-\sqrt{2}$, $y = 1-\left(2-\sqrt{2}\right) = -1+\sqrt{2}$

When $x = 2+\sqrt{2}$, $y = 1-\left(2+\sqrt{2}\right) = -1-\sqrt{2}$

</td><td>

Many students stopped at equation (2) and incorrectly concluded that it was the required equation of tangent.

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</table>

Qn		Remarks
2(ii)	$2-2y\dfrac{\mathrm{d}y}{\mathrm{d}x}=2(x+y)\left(1+\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$ $2=2(x+y)+2(x+2y)\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $2=2(x+y)+2(x+2y)\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1-(x+y)}{x+2y}$ When $x=0,\ y=0$, $-\dfrac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}=0$. Hence normal to C at the origin is $y=0$. When $x=2,\ y=-2$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{-2}$ Tangent to C at $A(2,-2)$, $y-(-2)=-\dfrac{1}{2}(x-2)$ Where the normal and the tangent intersect, $2=-\dfrac{1}{2}(x-2)$ $x=-2$ Area of triangle $OAB=\dfrac{1}{2}(2)(2)=2\text{ units}^2$	This part was badly done as many wrote the equation of normal to C as $x=0$, instead of $y=0$. Due to this error, they were unable to obtain the required area.
3(i)	Since a is real, the polynomial equation has real coefficients, and thus all non-real roots must be in conjugate pairs. Since the degree of the polynomial is three, there will be 3 roots. The highest even number below 3 is 2.	Candidates showed vague understanding of conjugate root theorem. Note that it does not imply that there WILL be complex conjugate roots.
3(ii)	$z^3+az^2+az+7=0$ $(-7)^3+a(-7)^2+a(-7)+7=0$ $a=8$ $z^3+8z^2+8z+7=0$ $(z+7)(z^2+z+1)=0$ $z=-7$ or $z=\dfrac{-1\pm \mathrm{i}\sqrt{3}}{2}$ $z=7e^{i\pi},\ \mathrm{e}^{\frac{i2\pi}{3}},\mathrm{e}^{-\frac{i2\pi}{3}}$	If candidates sub in $x=a+bi$ and $x=a-bi$ and proceed to solve for a and b, they will be penalised as the complex conjugate roots may not even exist in the first place.

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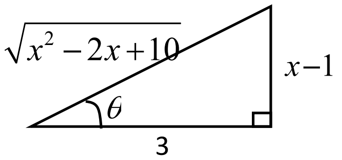
Qn		Remarks
3(iii)	$-iz^3-8z^2+8iz+7=0$ $(iz)^3+8(iz)^2+8(iz)+7=0$ From (ii), replace z with iz $iz=7e^{i\pi},\ e^{\frac{i2\pi}{3}},e^{-\frac{i2\pi}{3}}$ $\Rightarrow z=-i7e^{i\pi},\ -ie^{\frac{i2\pi}{3}},-ie^{-\frac{i2\pi}{3}}$ $\Rightarrow z=e^{-\frac{i\pi}{2}}7e^{i\pi},\ e^{-\frac{i\pi}{2}}e^{\frac{i2\pi}{3}},e^{-\frac{i\pi}{2}}e^{-\frac{i2\pi}{3}}$ $\Rightarrow z=7e^{\frac{i\pi}{2}},\ e^{\frac{i\pi}{6}},e^{\frac{i5\pi}{6}}$	
4(i)	$x-1=3\tan\theta$ $\frac{dx}{d\theta}=3\sec^2\theta$ $\int\frac{1}{\sqrt{x^2-2x+10}}\,dx=\int\frac{1}{\sqrt{(x-1)^2+3^2}}\,dx$ $=\int\frac{1}{\sqrt{(3\tan\theta)^2+3^2}}\cdot3\sec^2\theta\ d\theta$ $=\int\frac{1}{3\sec\theta}\cdot3\sec^2\theta\ d\theta$ $=\int\sec\theta\ d\theta$ $=\ln\lvert\sec\theta+\tan\theta\rvert+C$ $=\ln\left\lvert\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\rvert+C$	Many candidates omitted the square root in the denominator when doing the substitution or applying the trigo identity. Apply **$\tan^2\theta+1=\sec^2\theta$** Use the right angle triangle to give answer in terms of x. $x-1=3\tan\theta\Rightarrow\tan\theta=\frac{x-1}{3}$  Remember the modulus sign, arbitrary constant and rewrite expression in terms of x.

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Qn		Remarks
4(ii)	$x+3=\frac{1}{2}(2x-2)+4$ $\int \frac{x+3}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$ $=\int \frac{\frac{1}{2}(2x-2)+4}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$ $=\frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-2x+10}}\,\mathrm{d}x+\int \frac{4}{\sqrt{(x-1)^2+3^2}}\,\mathrm{d}x$ $=\frac{1}{2}\frac{\sqrt{x^2-2x+10}}{\frac{1}{2}}+4\int \frac{1}{\sqrt{(x-1)^2+3^2}}\,\mathrm{d}x$ $=\sqrt{x^2-2x+10}+4\ln\left\lvert\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right\rvert+C$	Standard form integral $\int \underbrace{(2x-2)}_{\mathrm{f}'(x)}\underbrace{(x^2-2x+10)}_{\mathrm{f}(x)}^{-\frac{1}{2}}\mathrm{d}x=\frac{(x^2-2x+10)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C$ Many students erroneously applied the formula in MF26 when the form of the integral is not the same. Ans in (i)
5(i)	$\mathrm{f}(r)-\mathrm{f}(r+1)=\frac{\sqrt{r}}{2\sqrt{r}+1}-\frac{\sqrt{r+1}}{2\sqrt{r+1}+1}$ $=\frac{\left(2\sqrt{r}\sqrt{r+1}+\sqrt{r}\right)-\left(2\sqrt{r+1}\sqrt{r}+\sqrt{r+1}\right)}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$ $=\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$ $\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}=\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r+1)\right]$ $=\mathrm{f}(1)-\mathrm{f}(2)$ $+\mathrm{f}(2)-\mathrm{f}(3)$ $+\ldots$ $+\mathrm{f}(n)-\mathrm{f}(n+1)$ $=\mathrm{f}(1)-\mathrm{f}(n+1)$ $=\frac{1}{3}-\frac{\sqrt{n+1}}{2\sqrt{n+1}+1}$	

Qn		Remarks
5(ii)	$\displaystyle\sum_{r=1}^{\infty}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$ $\displaystyle=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{\sqrt{n+1}}{2\sqrt{n+1}+1}\right)$ $\displaystyle=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2+\frac{1}{\sqrt{n+1}}}\right)$ $\displaystyle=-\frac{1}{6}$	Most students have the misconception that a fraction must go to 0 when the denominator goes to infinity. However, note that the numerator goes to infinity as well, thus the expression is indeterminate until further manipulation is done to get a clearer picture.
5(iii)	$\displaystyle\sum_{r=1}^{n}\frac{\sqrt{r+1}-\sqrt{r+2}}{\left(2\sqrt{r+1}+1\right)\left(2\sqrt{r+2}+1\right)}=\sum_{r=2}^{n+1}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$ $\displaystyle=\sum_{r=1}^{n+1}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}-\frac{1-\sqrt{2}}{3(2\sqrt{2}+1)}$ $\displaystyle=\frac{1}{3}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}-\frac{1-\sqrt{2}}{3(2\sqrt{2}+1)}$ $\displaystyle=\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}$ Need $\displaystyle\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}<-0.1$ (see table below) Using GC, least $n=57$	One may alternatively input the expression as a summation into the GC, but the calculation of the sum for each n takes much longer. Some students were not careful with the number of decimal places, thus unable to compare the value with -0.1.

n	$\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}$
56	-0.099797
57	-0.100043

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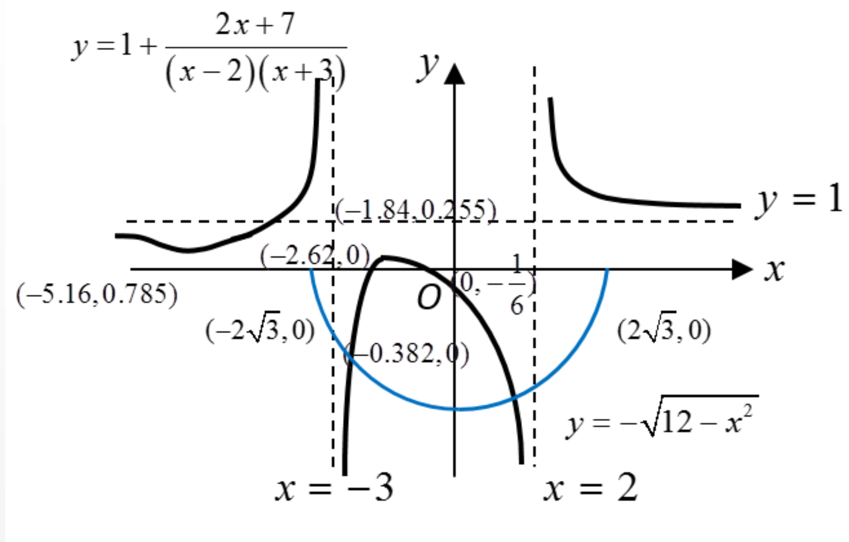
<table>
<tr><th>Qn</th><th></th><th>Remarks</th></tr>
<tr><td>6(i)</td><td>

$y = 1 + \dfrac{2x+p}{x^2+x-6}$
$\dfrac{dy}{dx} = \dfrac{2(x^2+x-6)-(2x+p)(2x+1)}{(x^2+x-6)^2}$
$\dfrac{dy}{dx} = 0 \Rightarrow 2(x^2+x-6)-(2x+p)(2x+1) = 0$
$2x^2+2px+12+p=0$
$4p^2-4(2)(12+p)>0$
$p^2-2p-24>0$
$(p+4)(p-6)>0$
$p<-4$ or $p>6$

</td><td>Learn to use quotient rule.</td></tr>
<tr><td>6(ii)</td><td>

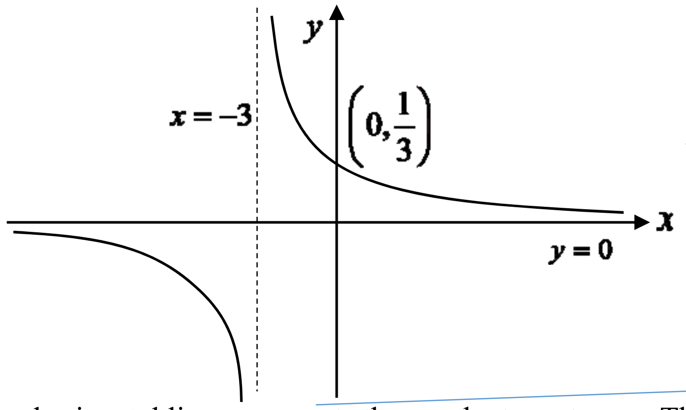
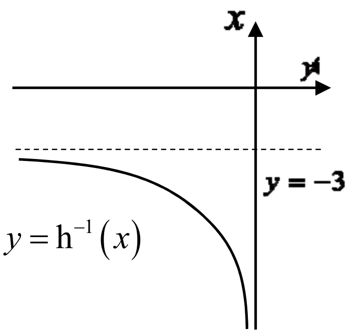
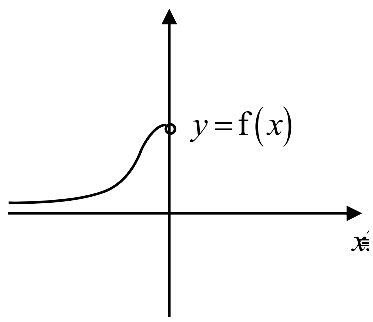


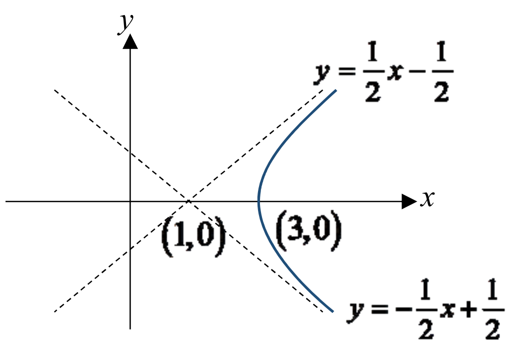
</td><td>Write down all the coordinates including the stationary points (in this case) as stated in the question.</td></tr>
<tr><td>6(iii)</td><td>

$1+\sqrt{12-x^2} \ge \dfrac{2x+7}{(2-x)(x+3)}$
$1+\dfrac{2x+7}{(x-2)(x+3)} \ge -\sqrt{12-x^2}$
Sketch $y=-\sqrt{12-x^2}$ (as above)

</td><td>Sketch $y=-\sqrt{12-x^2}$ on the same diagram with the end points touching the x-axis.

Label both graphs clearly using 2 different colours e.g. blue, black or pencil.</td></tr>
</table>

Qn		Remarks
	$-2\sqrt{3} \le x < -3$ OR $-2.92 \le x \le 1.46$ OR $2 < x \le 2\sqrt{3}$	
7(i)	 Every horizontal line $y = k$ cuts the graph at most once. This implies g is one-one. Therefore g^{-1} exists $\mathrm{g}^{-1} : x \mapsto \frac{1}{x} - 3, x \in \mathbb{R},\ x \ne 0$	There is a need to draw the graph to show that any horizontal line will cut the graph at most once. To show that function is 1-1, there is a need to have a general equation, $y = k$. This is different from "at only one point", as the line $y = 0$ does not cut the graph.
7(ii)	$\{x \in \mathbb{R} \mid x \ne 0\}$	Question asked for "solution set" therefore answer must be written in sets.
7(iii)	$\mathrm{R}_{\mathrm{g}^{-1}} = \mathrm{D}_{\mathrm{g}} = \mathbb{R} \setminus \{-3\}, \mathrm{D}_{\mathrm{f}} = \mathbb{R}^{-}$. Since $\mathrm{R}_{\mathrm{g}^{-1}} \not\subset \mathrm{D}_{\mathrm{f}}$, fg^{-1} does not exist.	
7(iv)	$k = -3$	
7(v) (a)	  $\mathrm{R}_{\mathrm{fh}^{-1}} = \left(0, \mathrm{e}^{-9}\right)$	

Qn		Remarks
7(iv) (b)	$\mathrm{h}(x)=\mathrm{h}^{-1}(x)$ $\mathrm{h}(x)=x$ $\dfrac{1}{x+3}=x$ $x^2+3x-1=0$ Since $x<-3$, $x=-3.30$ (3sf) Need to reject the other root which does not lie in to range $x<-3$.	
8(i)	$(x-1)^2=\left(\mathrm{e}^t+\mathrm{e}^{-t}\right)^2=\mathrm{e}^{2t}+2+\mathrm{e}^{-2t}$ $(2y)^2=\left(\mathrm{e}^t-\mathrm{e}^{-t}\right)^2=\mathrm{e}^{2t}-2+\mathrm{e}^{-2t}$ Hence $(x-1)^2-(2y)^2=4$ $\dfrac{(x-1)^2}{2^2}-y^2=1$ Alternative solution by students: $(x-1)+2y=2\mathrm{e}^t$ ___(1) $(x-1)-2y=2\mathrm{e}^{-t}$ ___(2) (1)×(2): $(x-1)^2-(2y)^2=4\mathrm{e}^t\mathrm{e}^{-t}=4$	
8(ii)		Many students simply drew the whole hyperbola given by the Cartesian equation in **(i)** without taking into account the original parametric equations. Note that one needs to check the range of values the x and y coordinates can take. For all $t\in\mathbb{R}$, $x=1+\mathrm{e}^t+\mathrm{e}^{-t}>0$. Just key in parametric equations in GC.

Qn		Remarks

8(iii)

When $x=3$, $3=1+\mathrm{e}^{t}+\mathrm{e}^{-t}$

$$\mathrm{e}^{t}+\mathrm{e}^{-t}=2$$

$$t=0$$

When $x=1+\mathrm{e}+\mathrm{e}^{-1}$, $t=\pm 1$ $\left(t=1:\ y>0,\ t=-1:\ y<0\right)$

$$x=1+\mathrm{e}^{t}+\mathrm{e}^{-t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^{t}-\mathrm{e}^{-t}$$

By symmetry

Note that the integral $\int y\mathrm{d}x$ refers to the area either above the x-axis (if $y>0$) or below the x-axis (if $y<0$)

Area of required region $=2\int_{3}^{1+\mathrm{e}+\mathrm{e}^{-1}} y\mathrm{d}x$

$$=2\int_{0}^{1}\frac{\mathrm{e}^{t}-\mathrm{e}^{-t}}{2}\left(\mathrm{e}^{t}-\mathrm{e}^{-t}\right)\mathrm{d}t$$

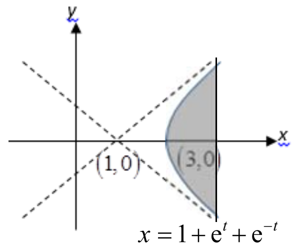
$$=\int_{0}^{1}\left(\mathrm{e}^{t}-\mathrm{e}^{-t}\right)^{2}\mathrm{d}t$$

$$=\int_{0}^{1}\left(\mathrm{e}^{2t}-2+\mathrm{e}^{-2t}\right)\mathrm{d}t$$

$$=\left[\frac{1}{2}\mathrm{e}^{2t}-2t-\frac{1}{2}\mathrm{e}^{-2t}\right]_{0}^{1}$$

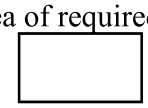
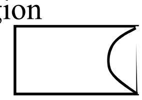
$$=\left[\frac{1}{2}\mathrm{e}^{2}-2-\frac{1}{2}\mathrm{e}^{-2}\right]-0$$

$$=\frac{1}{2}\left(\mathrm{e}^{2}-\mathrm{e}^{-2}\right)-2$$



<u>Alternatively (more tedious)</u>:

Area of required region

=  − 

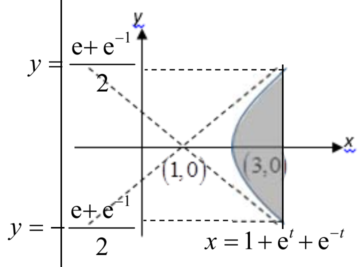
$$=\left(1+e+e^{-1}\right)\left(\frac{e-e^{-1}}{2}-\left(-\frac{e-e^{-1}}{2}\right)\right)-\int_{-\frac{\mathrm{e}-\mathrm{e}^{-1}}{2}}^{\frac{\mathrm{e}-\mathrm{e}^{-1}}{2}} x\ \mathrm{d}y$$

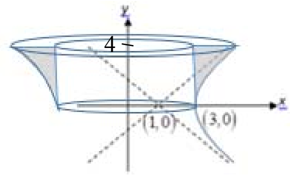
$$=\left(1+e+e^{-1}\right)\left(e-e^{-1}\right)-2\int_{0}^{\frac{\mathrm{e}-\mathrm{e}^{-1}}{2}} x\ \mathrm{d}y$$

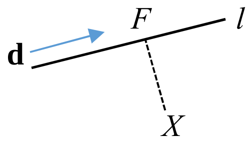
$$=\left(1+e+e^{-1}\right)\left(e-e^{-1}\right)-2\int_{0}^{1}\left(1+\mathrm{e}^{t}+\mathrm{e}^{-t}\right)\frac{\mathrm{e}^{t}+\mathrm{e}^{-t}}{2}\mathrm{d}t$$

$$\vdots$$

$$=\frac{1}{2}\left(\mathrm{e}^{2}-\mathrm{e}^{-2}\right)-2$$

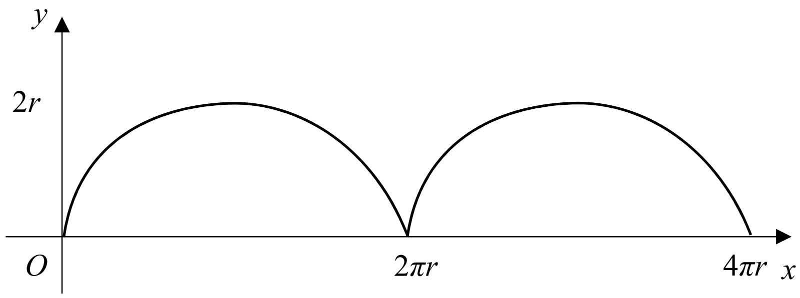


Qn		Remarks
8(iv)	$\dfrac{(x-1)^2}{2^2}-y^2=1$ $(x-1)^2=2^2\left(1+y^2\right)$ $x=1+2\sqrt{1+y^2}$ since $x>1$ Volume $=\pi\int_0^4 x^2\,\mathrm{d}y-\pi\left(3^2\right)(4)$ $=\pi\int_0^4\left(1+2\sqrt{1+y^2}\right)^2\mathrm{d}y-36\pi$ $=335$ units^3 (3 s.f.)	 Note that finding volume of revolution when the curve is defined parametrically is not in syllabus. Students can use the parametric equations to find volume but are not expected to do so. You should just use the Cartesian equation.
9(i)	$\overrightarrow{OP}=\begin{pmatrix}\lambda-\mu\\1+2\mu\\2-3\lambda\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-3\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\0\end{pmatrix}$ Locus of P is the plane with equation $\mathbf{r}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-3\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\0\end{pmatrix},\quad \lambda,\mu\in\mathbb{R}$	Students are expected to identify that the locus of P is a **plane** and to give the **correct equation** (in any acceptable form)
9(ii)	Normal of the locus of P (plane), $\begin{pmatrix}1\\0\\-3\end{pmatrix}\times\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}6\\3\\2\end{pmatrix}$ $\begin{pmatrix}6\\3\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\-3\end{pmatrix}=0$ Hence the line l and the plane are parallel. Equation of the plane, $\mathbf{r}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}=7$ $\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\2\\3\end{pmatrix}=8\neq 7$ Hence l is parallel to the plane and does not lie in the plane. Hence points P and Q will never meet. [Note that it is not sufficient just to show that l is parallel to the plane as it may actually lie on it. One must still need to show that there is a point on l that is not on the plane] Alternatively, one can check that	Many students made calculation error in the cross product. It is strongly advised that students **check the correctness** by either using the GC or simply taking the dot product between the resulting vector and any one of the two vectors and verify that it is zero. For e.g. $\begin{pmatrix}-1\\2\\0\end{pmatrix}\bullet\begin{pmatrix}6\\3\\2\end{pmatrix}=-6+6=0$ The marker pointed this out in the MY exam but apparently it has fallen on deaf ears

Qn		Remarks
	$\left[\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}\right]\bullet\begin{pmatrix}6\\3\\2\end{pmatrix}\neq 7$ for all $t\in\mathbb{R}$.	
9(iii)	Shortest distance between P and Q is the distance between the line and the parallel plane. $\dfrac{\left\lvert\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}-7\right\rvert}{\left\lvert\begin{pmatrix}6\\3\\2\end{pmatrix}\right\rvert}=\dfrac{2}{7}$	Many students used the wrong vector in this computation. Simply take a point on l, say (1,1,0) and compute its distance from the plane
9(iv)	Lines l and m are non-parallel. Hence $k\neq 1$. If the two lines intersect, $\begin{pmatrix}1\\k\\0\end{pmatrix}+s\begin{pmatrix}2\\-2\\-3k\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ for some $s,t\in\mathbb{R}$ $1+2s=1+2t$ ___(1) $k-2s=1-2t$ ___(2) $-3sk=-3t$ ___(3) From (1), $s=t$ Substituting $s=t$ in (2), $k=1$ $s=t$ and $k=1$ satisfies (3) Thus for the system of linear equations to be inconsistent, $k\neq 1$. Hence lines l and m are skew when $k\neq 1$.	Students must understand that if two lines are skew, then they are: ▪ Non-parallel ▪ Non-intersecting Students are expected to justify that $k\neq 1$ is the condition on k that satisfy the above two requirements
9(v)	Let F be the foot of perpendicular from X to line l. $\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ for some $t\in\mathbb{R}$ $\overrightarrow{XF}=\begin{pmatrix}1\\2\\5\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ $\left[\begin{pmatrix}1\\2\\5\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-2\\-3\end{pmatrix}=0$ $-17+17t=0$ $t=1$ $\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}2\\-2\\-3\end{pmatrix}=\begin{pmatrix}3\\-1\\-3\end{pmatrix}$ $F\equiv(3,-1,-3)$	Students must understand that XF and <u>not OF</u> is perpendicular to l.  Therefore $\overrightarrow{XF}\bullet\mathbf{d}=0$

Qn		Remarks
10	Let s be the distance between the front of the car and the train at time t. $s=\sqrt{x^2+30^2}$ and $x^2=(40-30t)^2+(20t)^2$ $s=\sqrt{(40-30t)^2+(20t)^2+30^2}=\sqrt{1300t^2-2400t+2500}$	
10(i)	$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600t-2400)$ When $t=1$, $\frac{\mathrm{d}s}{\mathrm{d}t}=2.67$	Generally ok. There are still handful of students not differentiating the expression given.
10(ii)	At stationary point $\frac{\mathrm{d}s}{\mathrm{d}t}=0$ $\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600t-2400)=0$ $2600t-2400=0\Rightarrow t=\frac{12}{13}$ $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600)$ $+(-\frac{1}{2})\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{3}{2}}(2600t-2400)$ When $t=\frac{12}{13},\frac{\mathrm{d}^2s}{dt^2}>0$ $s=\sqrt{1300(\frac{12}{13})^2-2400(\frac{12}{13})+2500}=$ 19.884=19.9	
10(iii)	Let the angle of elevation be θ $\sin\theta=\frac{30}{\sqrt{1300t^2-2400t+2500}}$ $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{1}{2}(30)(1300t^2-2400t+2500)^{-\frac{3}{2}}(2600t-2400)$ When $t=1$, $\cos\theta=\frac{\sqrt{(40-30)^2+20^2}}{\sqrt{1300-2400+2500}}=\sqrt{\frac{500}{1400}}$ $\therefore\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{1}{2}(30)(1300-2400+2500)^{-\frac{3}{2}}(200)\div\sqrt{\frac{500}{1400}}$ $=-0.0958$ rad/s (or -5.5°/s)	Most of the students are not able to form the trigo equation.

2017 NYJC JC2 Prelim 9758/1 Solution

Qn		Remarks
11(i)	 	Candidates often forget to indicate the axial intercepts. For those whom had written down the intercepts, many simply wrote 2π and 4π without the r
11(ii)	$\frac{dx}{d\theta}=r(1-\cos\theta)$, $\frac{dy}{d\theta}=r\sin\theta$ $\frac{dy}{dx}=\frac{\sin\theta}{1-\cos\theta}$ $=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ Use do formula Use trigo identity $=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=\cot\frac{\theta}{2}$	Some candidates did not make use of double angle formula and end up using a long and tedious method to arrive at the given answer
11(iii)	$\left(\frac{dy}{dx}\right)^2=\cot^2\frac{\theta}{2}=\operatorname{cosec}^2\frac{\theta}{2}-1$ $=\frac{1}{\sin^2\frac{\theta}{2}}-1$ Use double angle formula $=\frac{2}{1-\cos\theta}-1=\frac{2r}{r(1-\cos\theta)}-1$ Thus $\left(\frac{dy}{dx}\right)^2=\frac{2r}{y}-1$	Badly done with poor presentation. You are required to prove that $\left(\frac{dy}{dx}\right)^2=\frac{2r}{y}-1$. Many candidates started by stating this equation and proceed to prove. This is not true! The key is to apply double angle formula.
11(iv)	$Area=\int_0^{2\pi r} y\,dx$ $dx=\frac{dx}{d\theta}d\theta=r(1-\cos\theta)d\theta$ $=\int_0^{2\pi} r(1-\cos\theta)\cdot r(1-\cos\theta)\,d\theta$ $=r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$ Use double angle formula $=r^2\int_0^{2\pi}(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta)\,d\theta$ $=r^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$ $=3\pi r^2$	Many candidates are unable to write out the area expression in parametric form correctly. A handful of students did not integrate the expression correctly.

2017 NYJC Prelim Paper 2

Section A: Pure Mathematics [40 marks]

1 The position vectors of points A and B with respect to the origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. Point C lies on OA produced such that $4OA = AC$ and point D lies on OB produced such that $OB = BD$. The lines BC and AD meet at the point M.

(i) Giving a necessary condition for $\mathbf{a}$ and $\mathbf{b}$, find the position vector of M in terms of $\mathbf{a}$ and $\mathbf{b}$. [5]

(ii) If $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$, find the shortest distance of M from the line OC giving your answer in the form $k|\mathbf{a} \times \mathbf{b}|$ where k is a constant to be determined. [2]

2 **(a)** Find the set of values of θ lying in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ such that the sum to infinity of the geometric series $1 + \tan\theta + \tan^2\theta + \ldots$ is greater than 2. [5]

(b) The sum of the first n terms of a positive arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m. [4]

3 It is given that $y = \ln(\cos ax - \sin ax)$, where a is a non-zero constant.

(i) Show that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + a^2 = 0.$ [3]

(ii) By further differentiation of the result in **(i)**, find, in terms of a, the Maclaurin series for y, up to and including the term in x^3. [3]

(iii) Hence show that when x is small enough for powers of x higher than 2 to be neglected and $a = 2$, then $\cos 2x - \sin 2x \approx 1 + kx + kx^2$ where k is a constant to be determined. [4]

(iv) Using appropriate expansions from the List of Formulae (**MF26**), verify the correctness of your answer in **(iii)**. [2]

4 The growth of an organism in a controlled environment is monitored and the growth rate of the organism is proportional to $(N-x)x$, where x is the population (in thousands) of the organism at time t and N is a constant such that $x<N$. The initial population of the organism is $\frac{1}{3}N$.

(i) Find x in terms of t and determine the population of the organism in the long run, giving your answer in terms of N. [6]

Another model is proposed for the growth of the organism, which assumes the growth rate is purely a function of time and is modelled by the differential equation $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}=\frac{-9t}{\left(4+9t^2\right)^2}$. It predicts that the population of the organism will also eventually stabilise.

(ii) Show that under this model, $x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+\frac{N}{3}$.

Hence state the population of the organism in the long run, giving your answer in terms of N. [6]

Section B: Probability and Statistics [60 marks]

5 From past records, the number of days of hospitalization for an individual with minor ailment can be modelled by a discrete random variable with probability density function given by

$$P(X=x)=\begin{cases}\frac{6-x}{15}, & \text{for } x=1,2,3,4,5,\\ 0, & \text{otherwise.}\end{cases}$$

An insurance policy pays \$100 per day for up to 3 days of hospitalization and \$25 per day of hospitalization thereafter.

(i) Calculate the expected payment for hospitalization for an individual under this policy. [4]

(ii) The insurance company will incur a loss if the total payout for 100 hospitalisation claims under this policy exceed \$24000. Using a suitable approximation, estimate the probability that the insurance company will incur a loss for 100 such claims. [4]

6 A teacher wants to randomly form two teams of 5 students from a group of 5 girls and 5 boys for a sports activity. Two of the girls, Ann and Alice, are selected as team leaders. Find the probability that one team has exactly 3 girls. [2]

The ten students are seated at a round table of 10. Find the probability that

(i) Ann and Alice are not seated together, [2]

(ii) no two of the remaining 3 girls are next to each other given that Ann and Alice are not seated together. [4]

7 In a large company, a small sample of n employees is obtained to find out their mode of transport to work. The number of employees who ride the train to work is denoted by R. Assume that R has the distribution $\mathrm{B}(n, p)$.

(i) Given that $n = 10$, find the value of p if the probability that 6 employees ride the train to work is twice the probability that 4 employees ride the train to work. [3]

(ii) Given that $p = 0.25$, find the largest value of n such that the probability that fewer than 2 employees who ride the train to work is more than 0.15. [3]

(iii) Given that $n = 11$ and $p = 0.7$, find the probability that at least 5 employees ride the train to work if at least 3 employees do not ride the train to work. [4]

8 **(a)** Comment briefly on the following statements:

(i) Flowers in a garden are watered and the product moment correlation coefficient between petal size and the amount of water given is 0.073, so it follows that there is no relation between petal size and quantity of water given to the flower. [1]

(ii) The product moment correlation coefficient between the risk of heart disease and amount of red wine intake is found to be approximately -1. Therefore we conclude that red wine intake causes the risk of heart disease to decrease. [1]

(b) The median age of residents in Singapore across the years are given in the table.

Year (x)	**1984**	**1988**	**1992**	**1996**	**2000**	**2004**	**2008**	**2012**	**2016**
Median age (y)	26.7	28.8	27.0	32.3	34.0	35.4	36.7	38.4	40.0

It is thought that the median age of residents in year x can be modelled by one of the formulae

$$y = \frac{a}{x} + b, \quad y = c\ln x + d,$$

where a, b, c and d are constants.

(i) Plot a scatter diagram on graph paper for these values, labelling the axis, using a scale of 2cm to represent 5 years on the y-axis and an appropriate scale for the x-axis. One of the values of y was quoted wrongly. Indicate this point as P on your diagram. [2]

For parts **(ii)**, **(iii)**, **(iv)** of this question, you should **exclude** the point P.

(ii) Find, correct to 5 decimal places, the value of the product moment correlation coefficient between

(A) x^{-1} and y

(B) $\ln x$ and y. [2]

(iii) Explain which model is more appropriate to predict the median age of residents in Singapore and find the equation of the least squares regression line for this model, giving your answer to 2 decimal places. [2]

(iv) Explain why neither the regression line of x^{-1} on y nor the regression line of $\ln x$ on y should be used to estimate the year when the median age is 30. [1]

(v) Give a possible reason for the rise in the median age. [1]

9 A manufacturing process produces ball bearings with diameters with known standard deviation 0.04 cm. Under normal circumstances, the manufacturing process will produce ball bearings of mean diameter 0.5 cm.

(i) During a routine quality control check, a random sample of 25 ball bearings gives a mean of 0.51 cm. Is there evidence to believe that the manufacturing process is producing ball bearings of the stated diameter? Perform an appropriate test at 5% level of significance. State a necessary assumption for the test to be valid. [4]

An enhancement on the manufacturing process will ensure that the diameters of the ball bearings produced are less variable.

(ii) Measurements of a sample of 100 ball bearings give the following summary statistics:

$$\Sigma x = 50.6,\ \Sigma(x-0.5)^2 = 0.08345.$$

Show that the unbiased estimate of the population variance is 8.07×10^{-4}.
Is there evidence at the 5% level of significance that after the enhancement, the manufacturing process is producing oversized ball bearings? [4]

(iii) Another sample of 100 ball bearings yield the same summary statistics as the previous sample in **(ii)**. Explain, with justification, whether the combined sample will give a different conclusion to **(ii)**. [4]

10 The diameters of the bolts produced by two manufacturers A and B follow a normal distribution with a standard deviation of 0.16 mm.

The mean diameter of the bolts produced by manufacturer A is 1.56 mm. Of the bolts produced by manufacturer B, 24.2% have a diameter less than 1.52 mm.

(i) Show that the mean diameter of the bolts produced by manufacturer B is 1.632 mm. [3]

(ii) Find the probability that the diameter of a randomly chosen bolt from manufacturer A differs from the diameter of a randomly chosen bolt from manufacturer B by less than 0.1 mm. [3]

(iii) Find the probability that the total diameter of 5 randomly chosen bolt from manufacturer A is more than 5 times the diameter of a randomly chosen bolt from manufacturer B. [3]

(iv) A trading company buys 44% of its stock of bolts from manufacturer A and the rest from manufacturer B. A bolt is chosen at random from the trading company's stock. Show that the probability that the diameter of the bolt is less than 1.52 mm is 0.312. [3]

------END OF PAPER------

2017 NYJC JC2 Preliminary Examination 9758/2 Solution

Qn		Remarks
1(i)	Assume that $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. $\overrightarrow{OC}=5\mathbf{a}$, $\overrightarrow{OD}=2\mathbf{b}$ On the line BC, $\overrightarrow{OM}=\lambda(5\mathbf{a})+(1-\lambda)\mathbf{b}$ On the line AD, $\overrightarrow{OM}=\mu(2\mathbf{b})+(1-\mu)\mathbf{a}$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero, non-parallel vectors, comparing coefficient $\begin{matrix} 5\lambda=1-\mu \\ 2\mu=1-\lambda \end{matrix} \Rightarrow \lambda=\frac{1}{9}, \mu=\frac{4}{9}$ Thus $\overrightarrow{OM}=\frac{5}{9}\mathbf{a}+\frac{8}{9}\mathbf{b}$	Note that this is a condition on vectors, not points. You may use Ratio Theorem, or you may find direction vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$ to find the respective lines.
1(ii)	Since $\mathbf{a}$ is a unit vector in the direction of OC, shortest distance $=\left\|\overrightarrow{OM}\times\mathbf{a}\right\|$ $=\left\|\left(\frac{5}{9}\mathbf{a}+\frac{8}{9}\mathbf{b}\right)\times\mathbf{a}\right\|$ $=\frac{8}{9}\left\|\mathbf{a}\times\mathbf{b}\right\|$ $k=\frac{8}{9}$	Some students mixed up the properties for cross product and dot product. $\mathbf{a}\times\mathbf{a}=\mathbf{0}$, $\mathbf{a}\cdot\mathbf{a}=\left\|\mathbf{a}\right\|^2$ Note that distance cannot be negative. Always check your algebraic workings when your answers appear counter-intuitive.
2(a)	For sum to infinity to exist, $\left\|\tan\theta\right\|<1$ $-1<\tan\theta<1$ $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$ $\frac{1}{1-\tan\theta}>2$ $0<1-\tan\theta<\frac{1}{2}$ $\tan\theta>\frac{1}{2}\Rightarrow\theta>0.464$ Since $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$, therefore $\{\theta\in\mathbb{R} \mid 0.464<\theta<0.786\}$ or $\theta:(0.464, 0.786)$	Most students failed to check the range of values for $\left\|r\right\|<1$ for sum to infinity to exist. Note: Many students cross multiplied to get $1>2(1-\tan\theta)$. For this case it is ok as $1-\tan\theta>0$. **In general, we should not cross multiply for inequalities** unless the term multiplied is strictly positive. Set notation.

Qn		Remarks
2(b)	$u_1 = S_1 = 2 \Rightarrow a = 8$ $u_2 = S_2 - S_1 = 10 \Rightarrow d = 8$ $u_{32} = a + (32-1)d = 2 + (32-1)8 = 250$ $\dfrac{u_{32}}{u_m} = \dfrac{u_m}{u_2} = \text{constant}$ u_2, u_m and u_{32} are consecutive terms of GP $\Rightarrow (u_m)^2 = (10)(250) = 2500$ $u_m = 50$ (since it is a positive sequence) $50 = 2 + (m-1)8 \Rightarrow m = 7$ Alternatively, $u_n = S_n - S_{n-1}$ $= 4n^2 - 2n - \left[4(n-1)^2 - 2(n-1)\right]$ $= 8n - 6$ $\dfrac{u_{32}}{u_m} = \dfrac{u_m}{u_2}$ $\dfrac{8(32)-6}{8m-6} = \dfrac{8m-6}{8(2)-6}$ $(8m-6)^2 = (250)(10) = 2500$ $m = 7$ or $m = -5.5$ (rejected as m is a positive integer)	
3(i)	$y = \ln(\cos ax - \sin ax)$ $e^y = \cos ax - \sin ax$ $e^y \dfrac{dy}{dx} = -a \sin ax - a \cos ax$ $e^y \dfrac{d^2y}{dx^2} + e^y \left(\dfrac{dy}{dx}\right)^2 = -a^2 \cos ax + a^2 \sin ax$ $e^y \dfrac{d^2y}{dx^2} + e^y \left(\dfrac{dy}{dx}\right)^2 = -a^2 (\cos ax - \sin ax)$ $e^y \dfrac{d^2y}{dx^2} + e^y \left(\dfrac{dy}{dx}\right)^2 = -a^2 e^y$ $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + a^2 = 0$	A majority of students produced very long, tedious and messy calculations by direct differentiation when the calculation could have been much simpler by rewriting the equation into the implicit form $e^y = \cos ax - \sin ax$ and applying implicit differentiation to obtain the desired equation. Students MUST learn SMART WAYS of doing math instead of using the brute force method

Qn		Remarks
3(ii)	$\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=0$ When $x=0$, $y=0$ $\frac{dy}{dx}=-a$, $\frac{d^2y}{dx^2}=-2a^2$, $\frac{d^3y}{dx^3}=-4a^3$ $y=-ax-a^2x^2-\frac{2}{3}a^3x^3+\cdots$	Fairly straightforward application of Maclaurin's theorem to obtain series expansion.
3(iii)	$\ln\left(\cos 2x-\sin 2x\right)=-2x-4x^2-\frac{16}{3}x^3+\cdots$ $\cos 2x-\sin 2x\approx e^{-2x-4x^2}$ $\approx 1+\left(-2x-4x^2\right)+\frac{\left(-2x-4x^2\right)^2}{2!}$ (since $e^x\approx 1+x+\frac{x^2}{2!}$) $\approx 1-2x-4x^2+\frac{\left(-2x\right)^2}{2}$ $=1-2x-2x^2$ where $k=-2$	A number of students did not pay attention to the word 'Hence' which requires them to use an earlier result to deduce the next result. Many simply used the series expansions of sin x and cos x from MF26 which earn no credit
3(iv)	$\cos 2x-\sin 2x=1-\frac{(2x)^2}{2}-(2x)$ $=1-2x-2x^2$	Some students forgot the '2' and wrote $\cos 2x-\sin 2x$ $=1-\frac{x^2}{2}-x$ Some used the double angle formulae for sin $2x$ and cos $2x$ which is not necessary

Qn		Remarks
4(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=k(N-x)x$ $\frac{1}{(N-x)x}\frac{\mathrm{d}x}{\mathrm{d}t}=k$ $\int\frac{1}{(N-x)x}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $\frac{1}{N}\int\frac{1}{N-x}+\frac{1}{x}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $\frac{1}{N}\left(-\ln\lvert N-x\rvert+\ln\lvert x\rvert\right)=kt+C$ $\frac{1}{N}\ln\left\lvert\frac{x}{N-x}\right\rvert=kt+C$ $\ln\left\lvert\frac{x}{N-x}\right\rvert=Nkt+NC$ $\left\lvert\frac{x}{N-x}\right\rvert=\mathrm{e}^{Nkt+NC}$ $\frac{x}{N-x}=A\mathrm{e}^{Nkt}$ where $A=\pm\mathrm{e}^{NC}$ When $t=0$, $x=\frac{1}{3}N$, $A=\frac{1}{2}$ $\frac{x}{N-x}=\frac{1}{2}\mathrm{e}^{Nkt}$ $2x=(N-x)\mathrm{e}^{Nkt}$ $x\left(2+\mathrm{e}^{Nkt}\right)=N\mathrm{e}^{Nkt}$ $x=\frac{N\mathrm{e}^{Nkt}}{2+\mathrm{e}^{Nkt}}$ equivalently, $x=\frac{N}{2\mathrm{e}^{-Nkt}+1}$	Remember to include the modulus sign whenever the integral involves ln.
	<u>Alternative method</u> (not recommended):	

Qn		**Remarks**
	$\dfrac{1}{(N-x)x}\dfrac{\mathrm{d}x}{\mathrm{d}t}=k$ $-\displaystyle\int \dfrac{1}{x^2-Nx}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $-\displaystyle\int \dfrac{1}{\left(x-\frac{N}{2}\right)^2-\left(\frac{N}{2}\right)^2}\,\mathrm{d}x=\int k\,\mathrm{d}t$ $-\left(\dfrac{1}{2\left(\frac{N}{2}\right)}\ln\left\|\dfrac{\left(x-\frac{N}{2}\right)-\frac{N}{2}}{\left(x-\frac{N}{2}\right)+\frac{N}{2}}\right\|\right)=kt+C$ $-\left(\dfrac{1}{N}\ln\left\|\dfrac{x-N}{x}\right\|\right)=kt+C$ $\dfrac{1}{N}\ln\left\|\dfrac{x}{N-x}\right\|=kt+C$ $\ln\left(\dfrac{x}{N-x}\right)=Nkt+NC$ since $0<x<N$ When $t=0$, $x=\dfrac{1}{3}N$, $\ln\dfrac{1}{2}=NC \Rightarrow C=-\dfrac{1}{N}\ln 2$ $\ln\left(\dfrac{x}{N-x}\right)=Nkt-\ln 2$ $\ln\left(\dfrac{2x}{N-x}\right)=Nkt$ $\dfrac{2x}{N-x}=\mathrm{e}^{Nkt}$ $x\left(2+\mathrm{e}^{Nkt}\right)=N\,\mathrm{e}^{Nkt}$ $x=\dfrac{N\,\mathrm{e}^{Nkt}}{2+\mathrm{e}^{Nkt}}$ equivalently, $x=\dfrac{N}{2\,\mathrm{e}^{-Nkt}+1}$ As $t\to\infty$, $x=\dfrac{N}{2\mathrm{e}^{-Nkt}+1}\to N$	We can remove the modulus sign by taking into account the range of values the variable x can take

Qn		Remarks

(ii)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}=\frac{-9t}{\left(4+9t^2\right)^2}$$

$$\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\,\mathrm{d}t=\int \frac{-9t}{\left(4+9t^2\right)^2}\,\mathrm{d}t$$

$$=-\frac{1}{2}\int \frac{18t}{\left(4+9t^2\right)^2}\,\mathrm{d}t$$

Integral $\int \frac{18t}{\left(4+9t^2\right)^2}\,\mathrm{d}t$ is of the form $\int \mathrm{f}'(t)\left[\mathrm{f}(t)\right]^n \mathrm{d}t$ where $n=-2$.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{2}\left(\frac{1}{4+9t^2}\right)+A$$

$$\int \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t=\frac{1}{2}\int \frac{1}{4+9t^2}\,\mathrm{d}t+\int A\,\mathrm{d}t$$

$$=\frac{1}{18}\int \frac{1}{\left(\frac{2}{3}\right)^2+t^2}\,\mathrm{d}t+\int A\,\mathrm{d}t$$

$$=\frac{1}{18}\cdot\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{\frac{2}{3}}\right)+At+B$$

$$x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+At+B$$

Some students wrongly express the integral as $\frac{1}{2}\int \frac{1}{4+9t^2}+A\ \mathrm{d}t$.

Note that $\int \frac{1}{2}\left(\frac{1}{4+9t^2}\right)+A\ \mathrm{d}t=\frac{1}{2}\int \frac{1}{4+9t^2}+\underline{2}A\ \mathrm{d}t$.

Thus it is recommended to express as two separate integrals as shown in the solution.

To find $\int \frac{1}{4+9t^2}\ \mathrm{d}t$, it is necessary to have the coefficient of t^2 to be 1 before applying formula in MF26. Thus s

$$\int \frac{1}{4+9t^2}\ \mathrm{d}t=\frac{1}{9}\int \frac{1}{\left(\frac{2}{3}\right)^2+t^2}\ \mathrm{d}t=\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{\frac{2}{3}}\right)=\frac{3}{2}\tan^{-1}\left(\frac{3t}{2}\right)$$

and <u>not</u> $\int \frac{1}{4+9t^2}\ \mathrm{d}t=\int \frac{1}{2^2+(3t)^2}\ \mathrm{d}t=\frac{1}{2}\tan^{-1}\left(\frac{3t}{2}\right)$.

Since the population stabilises in the long run, as $t\to\infty$, $x\to$ finite value, $A=0$

When $t=0$, $x=\frac{1}{3}N$, $\quad B=\frac{N}{3}$

Hence $\quad x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+\frac{N}{3}$

When $t\to\infty$, $\tan^{-1}\left(\frac{3t}{2}\right)\to\frac{\pi}{2}$

Hence $x\to\frac{\pi}{24}+\frac{N}{3}$.

5(i)

Let Y be the payment for an individual. The probability table is as follows:

y	100	200	300	325	350
$P(Y=y)$	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{15}$

$$\mathrm{E}(Y)=100\cdot\frac{1}{3}+200\cdot\frac{4}{15}+300\cdot\frac{1}{5}+325\cdot\frac{2}{15}+350\cdot\frac{1}{15}$$

$$=213\frac{1}{3}$$

Remarks: Candidates must read the question carefully to understand the payment scale, and thereafter to write down the probability table for Y correctly. Note that there is no linear relationship between Y and X . Thus working out $E(X)$ will not obtain any credit.

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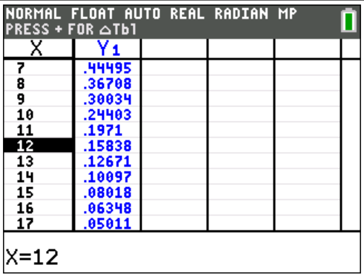
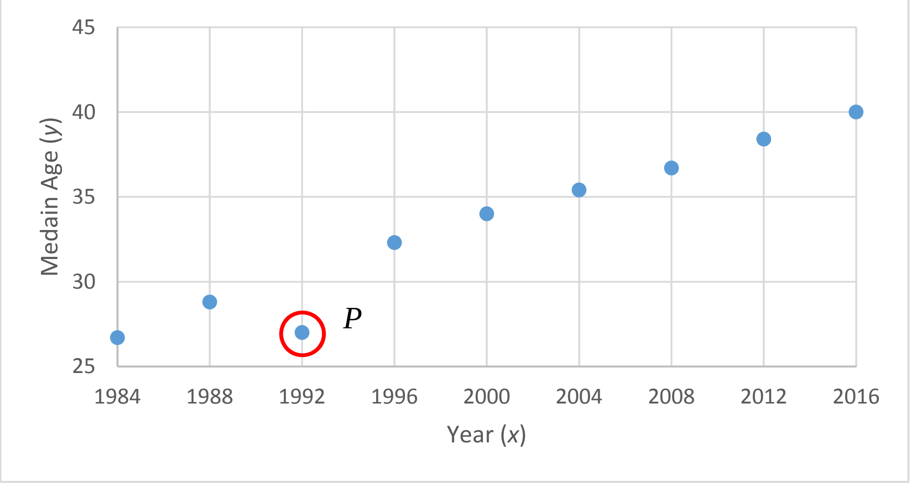
Qn		**Remarks**
5(ii)	$\mathrm{E}(Y^2)=100^2\cdot\frac{1}{3}+200^2\cdot\frac{4}{15}+300^2\cdot\frac{1}{5}+325^2\cdot\frac{2}{15}+350^2\cdot\frac{1}{15}$ $=54250$ $\mathrm{Var}(Y)=\mathrm{E}(Y^2)-\left[\mathrm{E}(Y)\right]^2=8738\frac{8}{9}$ Since $n=100$ is large, by Central Limit Theorem, $T=\sum_{i=1}^{100}Y_i\sim \mathrm{N}\left(21333.33, 873888.89\right)$ approx. Prob. Req'd $=P(T>24000)$ ≈ 0.00217	Note that $E(Y^2)$ is not the variance of Y. Further, candidates are reminded that you should not assume that Y is normally distributed. Central Limit Theorem is necessary to obtain the approximate distribution of ΣY_i.
6	Probability required = $\dfrac{2\times {}^3C_2\times {}^5C_2}{{}^8C_4}=\dfrac{6}{7}$	${}^3C_2\times {}^5C_2$ – choose 2 girls from the remaining 3 girls and 2 boys from 5 boys for the group with exactly 3 girls. Multiply by 2 because this group can be Ann or Alice's group. This is a conditional probability as Ann and Alice must be the team leaders, thus ${}^8C_4$.
6(i)	Required probability = $\dfrac{(8-1)!\,{}^8C_2\,2!}{(10-1)!}=\dfrac{7}{9}$ OR $\quad 1-\dfrac{(9-1)!\,2!}{(10-1)!}=\dfrac{7}{9}$	Insertion method or apply Principle of complementation

Qn		Remarks
6(ii)	Let X be the event that the remaining 3 girls are separated. Let Y be the event that Ann and Alice are not seated together. $\mathrm{P}(X \mid Y)=\dfrac{\mathrm{P}(X\cap Y)}{P(Y)}$ $=\dfrac{\mathrm{P}(X)-\mathrm{P}(X\cap Y')}{P(Y)}$ $=\dfrac{\dfrac{(7-1)!\,{}^{7}C_{3}\,3!-(6-1)!\,2!\,{}^{6}C_{3}\,3!}{(10-1)!}}{7/9}$ $=\dfrac{85/252}{7/9}$ $=\dfrac{85}{196}$	Not possible to list out all the cases.
7(i)	$\mathrm{P}(R=6)=2\mathrm{P}(R=4)$ ${}^{10}C_{6}\,p^{6}(1-p)^{4}=2\,{}^{10}C_{4}\,p^{4}(1-p)^{6}$ $p^{2}=2(1-p)^{2}$ $p^{2}-4p+2=0$ $p=0.586$	
7(ii)	$R\sim\mathrm{B}(n,0.25)$ $\mathrm{P}(R<2)>0.15$ $\mathrm{P}(R\le 1)>0.15$ $n=12$  NORMAL FLOAT AUTO REAL RADIAN MP PRESS + FOR △Tbl X: 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 Y1: .44495, .36708, .30034, .24403, .1971, .15838, .12671, .10097, .08018, .06348, .05011 X=12	

Qn		Remarks
7(iii)	$R \sim \mathrm{B}(11, 0.7)$ $\mathrm{P}(R \geq 5 \mid R \leq 8) = \dfrac{\mathrm{P}(R \geq 5 \text{ and } R \leq 8)}{\mathrm{P}(R \leq 8)}$ $= \dfrac{\mathrm{P}(5 \leq R \leq 8)}{\mathrm{P}(R \leq 8)}$ $= \dfrac{\mathrm{P}(R \leq 8) - \mathrm{P}(R \leq 4)}{\mathrm{P}(R \leq 8)}$ $= 0.969$	2 major errors seen in the students' scripts. 1. Failure to recognise that this is a conditional probability. 2. Failure to count the number of cases for the numerator.
8(a) (i)	The value of 0.073 indicates that there is a weak linear correlation between petal size and the amount of water but there could be some non-linear relation.	Quite a number of students state that there is still some weak linear correlation.
8(a) (ii)	The approximate value of -1 indicates that there is a strong negative linear correlation between the risk of heart disease and amount of red wine intake. It does not mean that red wine intake decreases the risk of heart disease.	
8(b) (i)		Wrong scale is used by a handful of students.
8(b) (ii)	For Model A, $r = -0.9985438 = 0.99854$ For Model B, $r = 0.9984431 = 0.99844$	Quite a number of students chose model B instead of A simply because r is positive.
8(b) (iii)	Model A as the $\lvert r \rvert$ value is closer to 1. The suitable regression line is $y = 848.24 - \dfrac{1629165.57}{x}$	
8(b) (iv)	This is because age (x) is the controlled variable	Badly answered. Students are not able to identify controlled variable.

Qn		Remarks
8(b)(v)	The rise in the median age is due to the drop in the growth of the population.	
9(i)	To test $H_0: \mu = 0.5$ $H_1: \mu \neq 0.5$ Two-tailed test as the question asked for the same diameter **or not.** Level of significance: 5% Under H_0, $Z = \dfrac{\bar{X} - 0.5}{0.04/\sqrt{25}} \sim N(0,1)$ Under H_0, we have assumed that $\mu = 0.5$. Thus we have to replace μ by 0.5. Note that the population variance is given for this part! Reject H_0 if p -value ≤ 0.05 Calculation: $\bar{x} = 0.51$, p -value = 0.211 Since p -value > 0.05, we do not reject H_0. Thus there is insufficient evidence at 5% level of significance that the manufacturing process is producing ball bearings of different diameters. Distribution of the diameter of the ball bearings is normal. As sample size is small.	On the whole, this question was badly done. Many candidates still showed poor presentation. They have also illustrated poor understanding in the writing of rejection region/criteria. Common mistakes include swapping μ_0 and $\bar{x}$; identify H_1 wrongly.
9(ii)	To test $H_0: \mu = 0.5$ $H_1: \mu > 0.5$ Level of significance: 5% Under H_0, by Central Limit Theorem, $Z = \dfrac{\bar{X} - 0.5}{s/\sqrt{100}} \sim N(0,1)$ approx Reject H_0 if p -value ≤ 0.05 Calculation: $\bar{x} = 0.506$, $\Sigma(x - 0.5) = 50.6 - 50 = 0.6$ We must obtain $\Sigma(x-0.5)$ first before applying formula $s^2 = \dfrac{1}{n-1}\left[\Sigma(x-0.5)^2 - \dfrac{(\Sigma(x-0.5))^2}{n}\right]$ $= \dfrac{1}{99}\left(0.08345 - \dfrac{0.6^2}{100}\right)$ $= 8.07 \times 10^{-4}$ Note that $\Sigma(x-0.5)^2$ is not sample variance as 0.5 is not the sample mean! p -value = 0.0173 Since p -value < 0.05, we reject H_0. Thus there is sufficient evidence at 5% level of significance that the manufacturing process is producing oversized ball bearings.	A handful of students identify this as a left tail test, resulting in a p -value that is more than 0.5. Note that for hypothesis testing, it does not make sense to have a p -value that is more than 0.5. As a rule of thumb, if $\bar{x} > \mu_0$, we should test $\mu > \mu_0$.

Qn		Remarks
9(iii)	For the new sample, $\overline{x}=0.506$. However, $s^2=\frac{1}{199}\left(2(0.08345)-\frac{\left(2(0.6)\right)^2}{200}\right)$ $=8.02513\times10^{-4}$ $Z_{calc}=2.995$, p-value $=0.00137$ Since p-value <0.05, we will still reject H_0. The conclusion remains the same. Combined sample is **NOT** pooled sample. Please do not use this formula at all! (It's for Further Math)	For this part, candidates are supposed to either compute the p-value after the samples are combined, or give clearly supported reason why the p-value will be smaller.
10(i)	Let X denotes the diameter of bolt from manufacturer A. $X\sim\mathrm{N}(1.56,0.16^2)$ Let Y denotes the diameter of bolt from manufacturer B. $Y\sim\mathrm{N}(\mu,0.16^2)$ $\mathrm{P}(Y<1.52)=0.242$ $\mathrm{P}(Z<\frac{1.52-\mu}{0.16})=0.242$ $\frac{1.52-\mu}{0.16}=-0.6998836\Rightarrow\mu=1.63198=1.632$	Candidates are expected to show full working for this part as it is a 'show' question. Standardising is the preferred method. Candidates who used graphical method using did not explain the graphs used and how the final answer is attained. Tables should not be used when dealing with a non-integer value.
10(ii)	$W=X-Y\sim\mathrm{N}(1.56-1.632,0.16^2+0.16^2)$ $\mathrm{P}(\lvert W\rvert<0.1)=\mathrm{P}(-0.1<W<0.1)=0.326$	Quite a few candidates find the distribution of $\lvert W\rvert$ instead of W, which is conceptually incorrect, and hence leading to a wrong answer.
10(iii)	$X_1+X_2+X_3+X_4+X_5-5Y\sim\mathrm{N}(5(1.56)\text{-}5(1.632),\ 5(0.16^2)+5^2(0.16)^2)$ $\mathrm{P}(X_1+X_2+X_3+X_4+X_5>5Y)$ $=\mathrm{P}(X_1+X_2+X_3+X_4+X_5-5Y>0)$ $=0.341$	Mistakes on calculating the correct variance were not as common this time round as compared to Midyear exams. However, poor representation of the variables is still commonly seen.
10(iv)	$\mathrm{P}(X<1.52)=0.40129$ $\text{Prob. Req'd}=(0.44)(0.4012)+0.56(0.242)$ $=0.3120876$ $=0.312$	Many candidates were unable to tackle this part. Again, as this is a 'show' question, candidates are expected to work out $\mathrm{P}(X<1.52)=0.40129$.