Additional Mathematics (4047) Paper 1 - ANSWERS

1 **(i)** $\cos\theta = -\dfrac{1}{\sqrt{1+a^2}}$

(ii) $\operatorname{cosec}\,\theta = -\dfrac{\sqrt{1+a^2}}{a}$

2 $2-2\sqrt{2} < k < 2+2\sqrt{2}$

3 $h=-1, k=\dfrac{1}{4}$

4 $\dfrac{6}{5\pi}$ cm/s

5 **(i)** \$150 000
(ii) \$103413
(iii) 99 months

6 **(i)** $(2x+3)(x+2)(x-2)$

(ii) $x=-3, x=-4$ or $x=4$

7 **(i)** $1-\dfrac{16}{x}+\dfrac{112}{x^2}+.......$

(ii) Constant term $=19$

8 **(i)** $\dfrac{dy}{dx}=-1+\dfrac{9}{(x-3)^2}$,

$\dfrac{d^2y}{dx^2}=-\dfrac{18}{(x-3)^3}$

(ii) 0 and 6

(iii) $\dfrac{d^2y}{dx^2}=\dfrac{2}{3}>0$, minimum point at $x=0$

$\dfrac{d^2y}{dx^2}=-\dfrac{2}{3}<0$, minimum point at $x=6$

9 **(i)** $k=6$

(ii) $\left(0, 14\dfrac{1}{4}\right)$

(iii) 65 units^2

10 **(a)** $x=-2\dfrac{2}{3}, y=-1\dfrac{2}{3}$

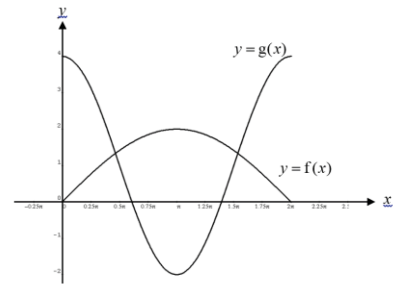
(b) $\dfrac{7\left(43+8\sqrt{5}\right)}{12}$ cm^3

11 **(i)** $t=2$ and $t=3$

(ii) $4\dfrac{1}{2}$ m

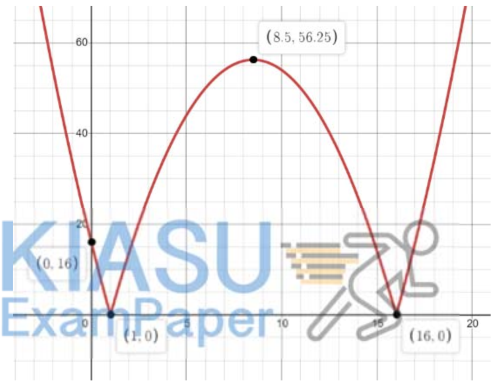
(iii) $4\dfrac{5}{6}$ m

12 **(i)** 4π
(ii) 0
(iii) 4
(iv) 4
(v)



(vi) $a \le x \le b$

Answer Key

1(ii)	$8x^2 - 335x - 343 = 0$	**8(ii)**	$x < -2$, $7 < x < 10$ and $x > 19$
2(a)	$\cot A = \frac{17}{6}$	**8(iii)**	$0 < k \leq 56\frac{1}{4}$
2(bii)	$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$	**8(iv)**	**3**
3(i)	coordinates of A = (4, 2) radius of C_1 = 3 units	**9(ii)**	$h = 30\sqrt{34}\cos(\theta - 1.03)$
3(iii)	7 units	**9(iii)**	Greatest value of $h = 30\sqrt{34}$ when $\theta = 1.03$.
3(iv)	4 or 10	**9(iv)**	$\theta = 0.614$ or 1.45
4(a)	2^{n-3m}	**10(ii)**	A =75.6 (accept 73.7 to 77.5) $k = 0.11$ (accept 0.105 to 0.115)
4(b)	$x = \frac{12}{5}$	**10(iii)**	90.6°C (accept 88.7 to 92.5)
5(ii)	$2666\frac{2}{3}\pi$ cm^3	**10(iv)**	8.25 minutes (accept 8 to 8.5 minutes)
6(i)	$-\frac{3}{x-2} + \frac{2}{x+2} + \frac{4x}{2x^2+1}$		
6(ii)	$\frac{4x}{2x^2+1}$		
6(iii)	$-3\ln(x-2) + 2\ln(x+2) + \ln(2x^2+1) + c$		
7(i)	Coordinates of $C = (-\frac{7}{6}, 0)$		
7(ii)	31.25 units^2		
8(i)			

O	ANDERSON SECONDARY SCHOOL Preliminary Examination 2020 Secondary Four Express & Five Normal

CANDIDATE NAME:

CLASS: / INDEX NUMBER:

ADDITIONAL MATHEMATICS

4047/01

Paper 1

MARKING SCHEME

4 August 2020

2 hours

0800 – 1000h

Candidates answer on the Question Paper.

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid/tape.

Answer **all** the questions.
Omission of essential working will result in loss of marks.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the test, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 80.

Setter: Mr Wong Teck Hock

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{r} a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ \ldots\ (n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 Given that θ is obtuse and $\tan\theta = a$, express, in terms of a,

(i) $\cos\theta$, [3]

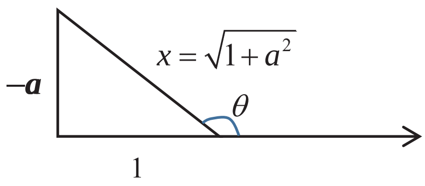
(ii) $\operatorname{cosec}\theta$. [2]

Solution:

(i) Since $\tan\theta = a$ and θ is obtuse, $a < 0$. **M1**

$x = \sqrt{1+a^2}$ **M1**

$\cos\theta = -\dfrac{1}{\sqrt{1+a^2}}$ **A1**



(ii) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$= \dfrac{1}{\dfrac{-a}{\sqrt{1+a^2}}}$ **M1**

$= -\dfrac{\sqrt{1+a^2}}{a}$ **A1**

2 Find the set of values of the constant k for which the curve $y = -x^2 + (1-k)x - 2$ lies entirely below the line $x + y = 0$. [4]

Solution:

$x + y = 0$

$y = -x$ -------- (1)

$y = -x^2 + (1-k)x - 2$ -------- (2)

$-x > -x^2 + (1-k)x - 2$

$x^2 + (k-2)x + 2 > 0$ for all real values of x **M1**

Discriminant < 0

$(k-2)^2 - 4(1)(2) < 0$ **M1**

$(k-2)^2 - 8 < 0$

$\left(k - 2 + \sqrt{8}\right)\left(k - 2 - \sqrt{8}\right) < 0$ **M1**

$\left(k - \left(2 - 2\sqrt{2}\right)\right)\left(k - \left(2 + 2\sqrt{2}\right)\right) < 0$

$\therefore 2 - 2\sqrt{2} < k < 2 + 2\sqrt{2}$ **A1**

(Accept $-0.828 < k < 4.83$)

3 Given that $y = he^x + \frac{k}{e^{2x}}$, and that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x + 2e^{-2x}$, find the value of each of the constants h and k. [4]

Solution:

$$y = he^x + \frac{k}{e^{2x}}$$

$$\frac{dy}{dx} = he^x - \frac{2k}{e^{2x}} \qquad \textbf{M1}$$

$$\frac{d^2y}{dx^2} = he^x + \frac{4k}{e^{2x}} \qquad \textbf{M1}$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = he^x + \frac{4k}{e^{2x}} - 2\left(he^x - \frac{2k}{e^{2x}}\right)$$

$$= -he^x + \frac{8k}{e^{2x}} \qquad \textbf{M1}$$

$$\therefore h = -1, k = \frac{1}{4} \qquad \textbf{A1}$$

4 A cylindrical ice block of base radius r cm is melting in such a way that the total surface area, A cm^2, is decreasing at a constant rate of 72 cm^2/s. Given that the height is twice the radius and assuming that the ice block retains its shape, calculate the rate of change of r when $r = 5$. [4]

Solution:

Let total surface area of ice block be A cm^2 and $A = 2\pi r^2 + 2\pi r(2r)$ where r cm is the radius at any instance.

$$A = 2\pi r^2 + 2\pi r(2r) \qquad \textbf{M1}$$

$$= 6\pi r^2$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$= 12\pi r\frac{dr}{dt} \qquad \textbf{M1}$$

When $\frac{dA}{dt} = -72$ and $r = 5$,

$$-72 = 12\pi \times 5\frac{dr}{dt} \qquad \textbf{M1}$$

$$\frac{dr}{dt} = -\frac{72}{60\pi}$$

$$= -\frac{6}{5\pi} \qquad \textbf{A1}$$

$\therefore$ The radius of the ice block decreases at $\frac{6}{5\pi}$ cm/s when $r = 5$. (accept 0.382 cm/s)

5 Ms Lee bought her dream car last year. The value, $\$A$, of the car is given by the formula $A = 150\,000e^{-pt}$, where p is a constant and t is the age of the car in months. The value of the car after 2 years is expected to be $120 000.

(i) Find the amount which Ms Lee paid for the car. [1]

(ii) Determine the value of the car after 40 months. Give your answer to the nearest dollar. [3]

(iii) Find the age of the car when its value drops to $60 000. Give your answer to the nearest month. [2]

Solution:

(i) $A = 150\,000e^{-pt}$

When $t = 0, A = 150\,000$.

Ms Lee paid $150 000 for the car. **B1**

(ii) When $t = 24, A = 120\,000$.

$120\,000 = 150\,000e^{-24p}$

$e^{24p} = \frac{5}{4}$ **M1**

$24p = \ln\frac{5}{4}$

$p = \frac{\ln\frac{5}{4}}{24}$ **M1**

$= 0.0092977$

When $t = 40$,

$A = 150\,000e^{-40\times 0.0092977}$

$= 103412.65$

The expected value of the car after 40 months is <u>$103413</u> (nearest dollar) **A1**

(iii) $60000 = 150000e^{-0.0092977t}$

$e^{0.0092977t} = 2.5$ **M1**

$t = \frac{\ln 2.5}{0.0092977}$

$= 98.55$

The car will be 99 months old when its value drops to $60000. **A1**

6 **(i)** Factorise $2x^3+3x^2-8x-12$. [3]

Solution:

$$2x^3+3x^2-8x-12 = x^2(2x+3)-4(2x+3) \quad \textbf{M1}$$
$$= 2x(x^2-4)+3(x^2-4)$$
$$= (2x+3)(x^2-4) \quad \textbf{M1}$$
$$= (2x+3)(x+2)(x-2) \quad \textbf{A1}$$

Alternatively, let $f(x)=2x^3+3x^2-8x-12$.

$$f(2)=2(2^3)+3(2^2)-8(2)-12$$
$$=16-12-16-12$$
$$=0$$

$x-2$ is a factor of $f(x)$. **M1**

$$2x^3+3x^2-8x-12=(x-2)(2x^2+7x+6) \quad \textbf{M1}$$
$$=(x-2)(x+2)(2x+3) \quad \textbf{A1}$$

(ii) Hence solve the equation $\frac{x^3}{4}+\frac{3}{4}x^2-4x-12=0.$. [3]

Solution:

$$\frac{x^3}{4}+\frac{3}{4}x^2-4x-12=0$$
$$2\left(\frac{x}{2}\right)^3+3\left(\frac{x}{2}\right)^2-8\left(\frac{x}{2}\right)-12=0 \quad \textbf{M1}$$
$$\left(2\times\frac{x}{2}+3\right)\left(\frac{x}{2}+2\right)\left(\frac{x}{2}-2\right)=0 \quad \textbf{M1}$$
$$\underline{\underline{x=-3,\ x=-4 \text{ or } x=4}} \quad \textbf{A1}$$

7 **(i)** Write down and simplify the first three terms in the expansion, in descending powers of *x*, of $\left(1-\frac{2}{x}\right)^8$. [2]

(ii) Given that there is no *x* term in the expansion of $\left(1-2x-kx^2\right)\left(1-\frac{2}{x}\right)^8$, find the constant term in the expansion. [4]

Solution:

(i)
$$\left(1-\frac{2}{x}\right)^8 = 1+\binom{8}{1}\left(-\frac{2}{x}\right)+\binom{8}{2}\left(-\frac{2}{x}\right)^2+\ldots\ldots \quad \textbf{M1}$$
$$\underline{\underline{=1-\frac{16}{x}+\frac{112}{x^2}+\ldots.}} \quad \textbf{A1}$$

(ii) $\left(1-2x-kx^2\right)\left(1-\frac{2}{x}\right)^8 = \left(1-2x-kx^2\right)\left(1-\frac{16}{x}+\frac{112}{x^2}+....\right)$

$= -kx^2 - 2x + 16kx + 1 + 32 - 112k + ...$

$= -kx^2 + \left(16k-2\right)x + 33 - 112k + ...$ **M1 for coefficient of x**

Since there is no x term in the expansion, **M1 for constant term**

$16k - 2 = 0$

$k = \frac{1}{8}$ **M1**

Constant term $= 33 - \frac{112}{8}$

$= \underline{\underline{19}}$ **A1**

8 The equation of a curve is $y = 2 - x - \frac{2x+3}{x-3}$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. [3]

(ii) Find the x-coordinate(s) of the stationary point(s) of the curve. [3]

(iii) Determine the nature of each stationary point. [2]

Solution:

(i) $y = 2 - x - \frac{2x+3}{x-3}$

$\frac{dy}{dx} = -1 - \frac{\left(x-3\right)\times 2 - \left(2x+3\right)\times 1}{\left(x-3\right)^2}$ **M1**

$= -1 + \frac{9}{\left(x-3\right)^2}$ **A1**

$\frac{d^2y}{dx^2} = -\frac{18}{\left(x-3\right)^3}$ **A1**

(ii) At the stationary points, $\frac{dy}{dx} = 0$

$\frac{9-\left(x-3\right)^2}{\left(x-3\right)^2} = 0$ **M1**

$\left(x-3\right)^2 = 9$ **M1**

$x - 3 = \pm 3$

$x = 0$ or $x = 6$

x-coordinates of the stationary points of the curve are 0 and 6. **A1**

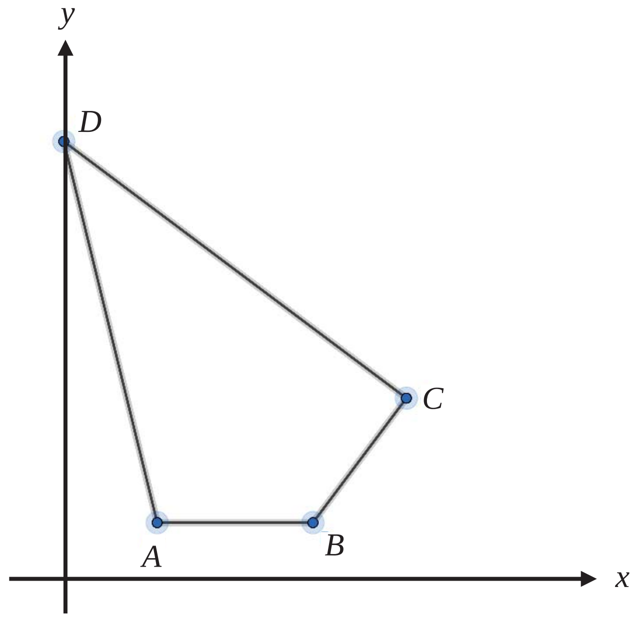
(iii) When $x = 0$, $\frac{d^2y}{dx^2} = \frac{2}{3} > 0$

The stationary point at $x = 0$ is a minimum point.

When $x = 6$, $\frac{d^2y}{dx^2} = -\frac{2}{3} < 0$ **M1A1**
(Correct value & correct conclusion

The stationary point at $x = 6$ is a maximum point.

9



The diagram shows a quadrilateral *ABCD* in which $AB = BC$, $\angle BCD = 90°$ and *D* is a point on the *y*-axis. The coordinates of the points *A*, *B* and *C* are $(3,2),(8,2)$ and $(11,k)$ respectively.

(i) Given that $k > 2$, show that $k = 6$. [3]

(ii) Find the coordinates of *D*. [3]

(iii) Find the length of *CD* and hence find the area of the quadrilateral *ABCD*. [5]

Solution:

(i) $AB = BC$

$\sqrt{(11-8)^2 + (k-2)^2} = 8-3$ **M1**

$9 + (k-2)^2 = 25$

$k - 2 = \pm 4$

$k = 2 \pm 4$ **M1**

$k = -2$ or $k = 6$

Since $k > 2$, $\underline{\underline{k = 6}}$ **A1**

(ii) Gradient of $BC = \dfrac{6-2}{11-8}$

$= \dfrac{4}{3}$

Gradient of $CD = -\dfrac{3}{4}$ **M1**

Let coordinates of *D* be $(0,h)$.

$\dfrac{h-6}{0-11} = -\dfrac{3}{4}$ **M1**

$h - 6 = -11\left(-\dfrac{3}{4}\right)$

$h = 14\dfrac{1}{4}$

Coordinates of $D = \underline{\underline{\left(0, 14\dfrac{1}{4}\right)}}$. **A1**

(iii) $CD=\sqrt{(0-11)^2+\left(14\frac{1}{4}-6\right)^2}$ units **M1**

$=13\frac{3}{4}$ units **A1**

M1

Area of $ABCD$ = area of ΔABD + area of ΔBCD

$=\frac{1}{2}\times5\times12\frac{1}{4}+\frac{1}{2}\times5\times13\frac{3}{4}$ units^2 **M1+M1**

$=\frac{5}{2}\left(12\frac{1}{4}+13\frac{3}{4}\right)$ units^2

$=\underline{\underline{65\text{ units}^2}}$ **A1**

10 **(a)** Find the values of x and y which satisfy the equations

$$\frac{1}{\sqrt{e^{2x-4y}}}=\frac{\sqrt[3]{e}}{e},$$

$$\frac{10^y}{2^x}=2\left(5^{x+1}\right).$$

[5]

Solution:

$\frac{1}{\sqrt{e^{2x-4y}}}=\frac{\sqrt[3]{e}}{e}$

$e^{2y-x}=e^{-\frac{2}{3}}$ **M1**

$2y-x=-\frac{2}{3}$

$2y=x-\frac{2}{3}$

$y=\frac{3x-2}{6}$ --------(1)

$\frac{10^y}{2^x}=2\left(5^{x+1}\right)$

$10^y=2\times5\times5^x\times2^x$

$10^y=10^{x+1}$ **M1**

$y=x+1$ ----------(2)

(1) = (2),

$\frac{3x-2}{6}=x+1$ **M1**

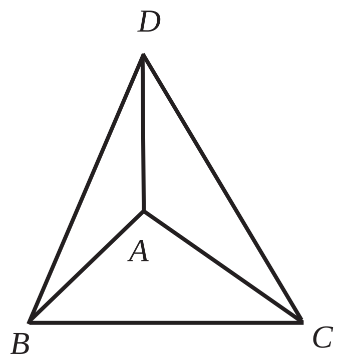
$\therefore 3x-2=6x+6$

$3x=-8$

A1+A1

$\underline{\underline{x=-2\frac{2}{3},\ y=-1\frac{2}{3}}}$

(b)



A solid triangular pyramid *ABCD,* with base *ABC* and vertex *D* such that *D* is vertically above *A*, has a base area of $\left(8+2\sqrt{5}\right)$ cm^2 and height $\left(12-\sqrt{5}\right)$cm . The top part of the pyramid is removed by a cut parallel to its base and passing through the midpoint of *AD*. Find the volume of the remaining solid, leaving your answer in the form $\dfrac{7\left(a+b\sqrt{5}\right)}{12}$cm^3, where *a* and *b* are integers. [4]

Solution:

$$\text{Volume of solid remaining} = \left(1-\left(\frac{1}{2}\right)^3\right)\times\frac{1}{3}\times\left(8+2\sqrt{5}\right)\times\left(12-\sqrt{5}\right)\ \text{cm}^2$$

Fraction of solid remaining M1
Volume of original pyramid M1

$$=\frac{7}{8}\times\frac{1}{3}\left(8+2\sqrt{5}\right)\times\left(12-\sqrt{5}\right)\ \text{cm}^2$$

$$=\frac{7\left(96-10+16\sqrt{5}\right)}{24}\ \text{cm}^2$$ **M1**

$$=\frac{7\left(43+8\sqrt{5}\right)}{12}\ \text{cm}^3$$ **A1**

11 A particle moving in a straight line passes a fixed point O with a velocity 6 ms^{-1}. The acceleration of the particle, a ms^{-2}, is given by $a = 2t - 5$, where t seconds is the time after passing O. Find

(i) the values of t when the particle is instantaneously at rest, [3]

(ii) the displacement of the particle from O at $t = 3$, [3]

(iii) the total distance travelled by the particle in the first 3 seconds of its motion. [2]

Solution:

(i) $a = 2t - 5$

Let $v = \int 2t - 5 \ \mathrm{d}t$

$= t^2 - 5t + c$ **M1**

When $t = 0, v = 6$.

$\therefore c = 6$

$v = t^2 - 5t + 6$ **M1**

When $v = 0,\ t^2 - 5t + 6 = 0$.

$(t-2)(t-3) = 0$

$t = 2$ or $t = 3$ **A1**

Particle is instantaneously at rest when $t = 2$ and $t = 3$

(ii) Let $s = \int t^2 - 5t + 6 \ \mathrm{d}t$

$= \frac{t^3}{3} - \frac{5t^2}{2} + 6t + c_1$ **M1**

When $t = 0, s = 0.\ \therefore c_1 = 0$.

$s = \frac{t^3}{3} - \frac{5t^2}{2} + 6t$ **M1**

When $t = 3$, $s = \frac{3^3}{3} - \frac{5 \times 3^2}{2} + 6 \times 3$

$= 9 - \frac{45}{2} + 18$

$= 4\frac{1}{2}$

Displacement of particle from O at $t = 3$ is $4\frac{1}{2}$ m **A1**

(iii) When $t = 2, s = \frac{2^3}{3} - \frac{5 \times 2^2}{2} + 6 \times 2$

$= \frac{8}{3} - 10 + 12$

$= 4\frac{2}{3}$ **M1**

Distance travelled in the first 3 seconds = $4\frac{2}{3} + 4\frac{2}{3} - 4\frac{1}{2}$ m

$= 4\frac{5}{6}$ m **A1**

12 It is given that $f(x)=2\sin\frac{x}{2}$ and $g(x)=3\cos x+1$ where $0\le x\le 2\pi$.

(i) State the period of $f(x)$. [1]

Solution:

Period = $\underline{\underline{4\pi}}$ **B1**

(ii) State the smallest value of $f(x)$. [1]

Solution:

Smallest value of $f(x)$ is $\underline{\underline{0}}$. **B1**

(iii) State the largest value of $g(x)$. [1]

Solution:

Largest value of $g(x)$ is $\underline{\underline{4}}$. **B1**

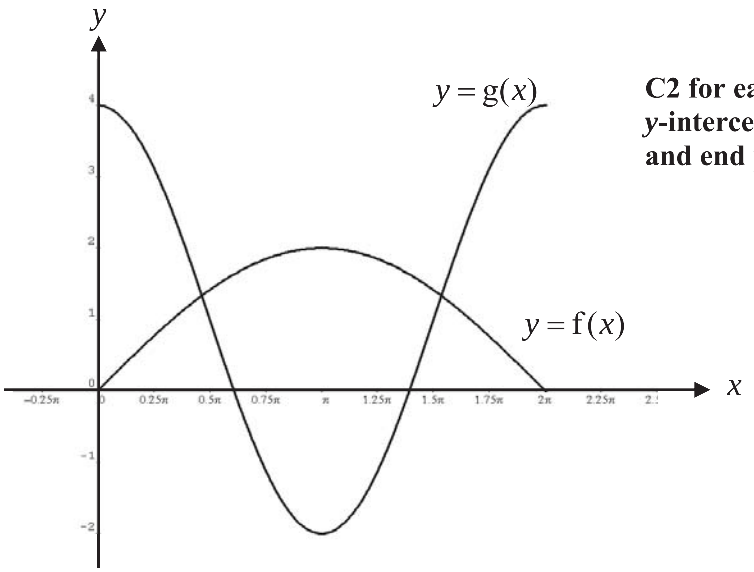
(iv) State the largest value of $|f(x)-g(x)|$. [1]

Solution:

Largest value of $|f(x)-g(x)|$ is $\underline{\underline{4}}$ **B1**

(v) Sketch, on the same axes, the graphs of $y=f(x)$ and $y=g(x)$ for $0\le x\le 2\pi$. [4]

Solution:



C2 for each graph (correct *y*-intercept, turning points and end points)

(vi) Given that the solutions to the equation $f(x)=g(x)$ for $0\le x\le 2\pi$ are a and b where $a<b$, state the range of value of x for which $f(x)\ge g(x)$. [1]

Solution:

$f(x)\ge g(x)$ when $\underline{\underline{a\le x\le b}}$ **B1**

Marking Scheme Prelim 2020 AM P2

1 The roots of the quadratic equation $5x-2x^2=2-k$ are α and β. The roots differ by $4\frac{1}{2}$.

(i) Show that $k=9$. [4]

$2x^2-5x+(2-k)=0$

Let α be the larger root.

Sum of roots: $\alpha+\beta=\frac{5}{2}$ **[M1]**

Product of roots: $\alpha\beta=\frac{2-k}{2}$ **[M1]**

Since roots differ by $4\frac{1}{2}$, $\alpha-\beta=\frac{9}{2}$

$(\alpha+\beta)^2=(\alpha-\beta)^2+4\alpha\beta$

$\left(\frac{5}{2}\right)^2=\left(\frac{9}{2}\right)^2+4\left(\frac{2-k}{2}\right)$ **[M1]**

$\frac{25}{4}=\frac{81}{4}+4-2k$

$2k=18$

$k=9$ (shown) **[A1]**

Alternative:

Let α be the larger root.

Sum of roots: $\alpha+\beta=\frac{5}{2}$ -- (1) **[M1]**

Since roots differ by $4\frac{1}{2}$, $\alpha-\beta=\frac{9}{2}$ -- (2)

(1) + (2): $2\alpha=7$

$\alpha=\frac{7}{2}, \beta=-1$ **[M1]**

Product of roots:

$\frac{2-k}{2}=(-1)\left(\frac{7}{2}\right)$ **[M1]**

$2-k=-7$

$k=9$ (shown) **[A1]**

(ii) Hence find a quadratic equation with integer coefficients whose roots are α^3 and β^3. [4]

Sum of roots:

$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$ **[M1]**

$=(\alpha+\beta)[(\alpha-\beta)^2+\alpha\beta]$

$=\frac{5}{2}\left[\left(\frac{9}{2}\right)^2+\frac{2-9}{2}\right]$

$=\frac{335}{8}$ **[M1]**

Product of roots:

$\alpha^3\beta^3=\left(-\frac{7}{2}\right)^3$

$=-\frac{343}{8}$ **[M1]**

The equation is $8x^2-335x-343=0$ **[A1]**

(accept any equivalent equation with integers coefficients)

Alternative:

Sum of roots:

$\alpha^3+\beta^3=(-1)^3(3.5)^3$ **[M1]**

$=\frac{335}{8}$ **[M1]**

Product of roots:

$\alpha^3\beta^3=\left(-1\times\frac{7}{2}\right)^3$

$=-\frac{343}{8}$ **[M1]**

The equation is

$8x^2-335x-343=0$ **[A1]**

2 **(a)** It is given that $\tan(A+B)=8$ and $\tan B=2$. **Without using a calculator**, find the exact value of $\cot A$. [3]

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

Since $\tan(A+B)=8$ and $\tan B=2$,

$$8=\frac{\tan A+2}{1-2\tan A}$$ **[M1]**

$8-16\tan A=\tan A+2$

$17\tan A=6$

$$\tan A=\frac{6}{17}$$ **[M1]**

$$\cot A=\frac{17}{6}$$ **[A1]**

(b) **(i)** Prove that $\sin 2x(\cot x-\tan x)=2\cos 2x$. [3]

$\text{LHS}=\sin 2x(\cot x-\tan x)$

$=2\sin x\cos x\left(\dfrac{\cos x}{\sin x}-\dfrac{\sin x}{\cos x}\right)$ **[M1]**

$=2(\cos^2 x-\sin^2 x)$ **[M1]**

$=2\cos 2x$ **[A1]**

$=\text{RHS}$ (proven)

(ii) Hence solve the equation $\sin 2x(\cot x-\tan x)=\sec 2x$ for $0\le x\le \pi$. [4]

$\sin 2x(\cot x-\tan x)=\sec 2x$

$$2\cos 2x=\frac{1}{\cos 2x}$$

$$\cos^2 2x=\frac{1}{2}$$ **[M1]**

$$\cos 2x=\pm\sqrt{\frac{1}{2}}$$ **[M1]**

Basic angle $=\dfrac{\pi}{4}$ rad

$$2x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$

[M1]

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$ **[A1]**

3 The equation of circle C_1, with centre A, is $x^2 + y^2 - 8x - 4y + 11 = 0$.

(i) Find the coordinates of A and the radius of C_1. [3]

$x^2 + y^2 - 8x - 4y + 11 = 0$
$(x-4)^2 - 16 + (y-2)^2 - 4 + 11 = 0$

$(x-4)^2 + (y-2)^2 = 9$ **[M1]**

coordinates of A = (4, 2) **[A1]**

radius of C_1 = 3 units **[A1]**

(ii) Explain why the line $x = 1$ is a tangent to C_1. [1]

Since the leftmost coordinates of the circle $= (1,\ 2)$, $x = 1$ is a tangent to C_1.
[B1]

The equation of a second circle C_2, with centre B, is $(x+3)^2 + (y-2)^2 = R^2$.

(iii) Find the length of AB. [2]

coordinates of B = $(-3,\ 2)$ **[M1]**

length of $AB = 7$ units **[A1]**

(iv) State the possible value(s) of R such that C_1 and C_2 touch each other at exactly one point. [2]

$R = 7 - 3 = 4$ **[B1]** or $R = 7 + 3 = 10$ **[B1]**

4 **(a)** Given that $\log_{\sqrt{2}} x = m$ and $\log_4 y = n$, express $\frac{\sqrt{y}}{x^6}$ in terms of m and n. [4]

$\log_{\sqrt{2}} x = m$

$x = \left(\sqrt{2}\right)^m$ --- (1)

$\log_4 y = n$

$y = 4^n$ --- (2)

[M1 for both (1) and (2)**]**

$$\frac{\sqrt{y}}{x^6} = \frac{\sqrt{4^n}}{\left(\sqrt{2}\right)^{6m}} \quad \textbf{[M1]}$$

$$= \frac{2^n}{2^{3m}} \quad \textbf{[M1]}$$

$$= 2^{n-3m} \quad \textbf{[A1]}$$

(b) Solve the equation $\log_2(3x-2) = \log_4\left(x^2+1\right) + \frac{2}{\log_{\sqrt{2}} 2}$. [6]

$$\log_2(3x-2) = \log_4\left(x^2+1\right) + \frac{2}{\log_{\sqrt{2}} 2}$$

$$\log_2(3x-2) = \frac{\log_2\left(x^2+1\right)}{\log_2 4} + \frac{2\log_2\sqrt{2}}{\log_2 2} \quad \textbf{[M1 change of base]}$$

$$\log_2(3x-2) = \frac{1}{2}\log_2(x^2+1) + 1 \quad \textbf{[M1]}$$

$$\log_2(3x-2) = \log_2(x^2+1)^{\frac{1}{2}} + \log_2 2$$

$$\log_2(3x-2) = \log_2 2(x^2+1)^{\frac{1}{2}} \quad \textbf{[M1]}$$

$$3x-2 = 2(x^2+1)^{\frac{1}{2}}$$

$$(3x-2)^2 = 4(x^2+1)$$

$$9x^2 - 12x + 4 = 4x^2 + 4$$

$$5x^2 - 12x = 0 \quad \textbf{[M1]}$$

$$x(5x-12) = 0$$

$x = 0$ (reject because $3x-2>0$) **[A1** for rejecting with reason**]**

or $x = \frac{12}{5}$ **[A1]**

Alternative:

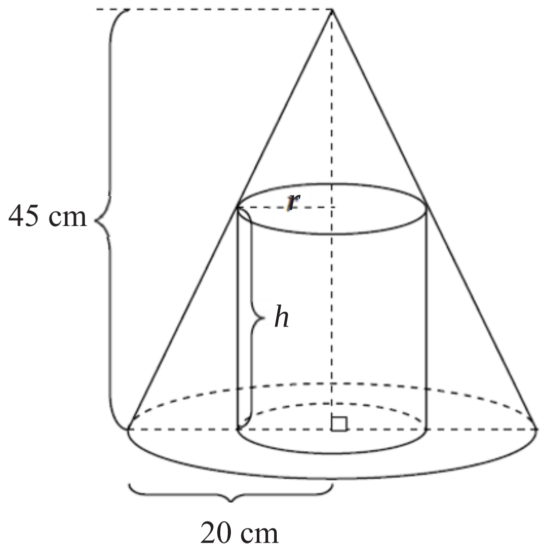
$$2\log_2(3x-2) - \log_2(x^2+1) = 2$$

$$\log_2(3x-2)^2 - \log_2(x^2+1) = 2$$

$$\log_2 \frac{(3x-2)^2}{x^2+1} = 2 \quad \textbf{[M1]}$$

$$\frac{(3x-2)^2}{x^2+1} = 4$$

5 The diagram shows a solid cylinder of radius r cm and height h cm inscribed in a hollow cone of height 45 cm and base radius 20 cm. The cylinder rests on the base of the cone and the circumference of the top surface of the cylinder touches the curved surface of the cone.



(i) Show that the volume, V cm^3, of the cylinder is given by $V = 45\pi r^2 - \frac{9}{4}\pi r^3$. [3]

By similar triangles,

$$\frac{45-h}{45} = \frac{r}{20} \quad \textbf{[M1]}$$

$$20(45-h) = 45r$$

$$900 - 20h = 45r$$

$$h = \frac{900-45r}{20}$$

$$= 45 - \frac{9}{4}r \quad \textbf{[M1]}$$

$$V = \pi r^2 h$$

$$= \pi r^2\left(45 - \frac{9}{4}r\right) \quad \textbf{[A1]}$$

$$= 45\pi r^2 - \frac{9}{4}\pi r^3 \text{ (shown)}$$

(ii) Given that r can vary, find the maximum volume of the cylinder, leaving your answer in terms of π. [4]

$$\frac{dV}{dr} = 90\pi r - \frac{27}{4}\pi r^2 \quad \textbf{[M1]}$$

When V is maximum, $\frac{dV}{dr} = 0$

$$90\pi r - \frac{27}{4}\pi r^2 = 0$$

$$\pi r\left(90 - \frac{27}{4}r\right) = 0$$

$r = \frac{40}{3}$ (since $r > 0$) **[M1]**

$$\frac{d^2V}{dr^2} = 90\pi - \frac{27}{2}\pi r$$

When $r = \frac{40}{3}$,

$\frac{d^2V}{dr^2} = 90\pi - \frac{27}{2}\pi\left(\frac{40}{3}\right) < 0$ **[M1]**

Since $\frac{d^2V}{dr^2} < 0$, volume of cylinder is maximum when $r = \frac{40}{3}$.

$$\text{maximum volume} = 45\pi\left(\frac{40}{3}\right)^2 - \frac{9}{4}\pi\left(\frac{40}{3}\right)^3$$

$= 2666\frac{2}{3}\pi \text{ cm}^3$ **[A1]**

(iii) Hence show that the cylinder occupies at most $\frac{4}{9}$ of the volume of the cone. [2]

$\text{volume of cone} = \frac{1}{3}\pi(20^2)(45)$

$= 6000\pi \text{ cm}^3$ **[M1]**

When the volume of cylinder is the maximum,

$$\frac{\text{volume of cylinder}}{\text{volume of cone}} = \frac{2666\frac{2}{3}\pi}{6000\pi}$$

$= \frac{4}{9}$ **[A1]**

Therefore, the cylinder occupies at most $\frac{4}{9}$ of the volume of the cone.

6 **(i)** Express $\dfrac{2x^3-20x^2-17x-10}{\left(x^2-4\right)\left(2x^2+1\right)}$ in partial fractions. [5]

$$\frac{2x^3-20x^2-17x-10}{\left(x^2-4\right)\left(2x^2+1\right)}=\frac{2x^3-20x^2-17x-10}{(x-2)(x+2)\left(2x^2+1\right)}$$

$$\frac{2x^3-20x^2-17x-10}{(x-2)(x+2)\left(2x^2+1\right)}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{2x^2+1} \quad \textbf{[M1]}$$

$$2x^3-20x^2-17x-10=A(x+2)\left(2x^2+1\right)+B(x-2)\left(2x^2+1\right)+(Cx+D)(x-2)(x+2)$$

Let $x=2$, $\quad -108=36A$

$A=-3$ **[M1]**

Let $x=-2$, $\quad -72=-36B$

$B=2$ **[M1]**

Let $x=0,\ A=-3,\ B=2$, $\quad -10=-6-4-4D$

$D=0$

Let $x=1,\ A=-3,\ B=2,\ D=0$, $\quad -45=-27-6-3C$

$C=4$ **[M1]**

Hence $\dfrac{2x^3-20x^2-17x-10}{\left(x^2-4\right)\left(2x^2+1\right)}=-\dfrac{3}{x-2}+\dfrac{2}{x+2}+\dfrac{4x}{2x^2+1}$ **[A1]**

(ii) Differentiate $\ln\left(2x^2+1\right)$ with respect to x. [2]

$$\frac{d}{dx}\left(\ln\left(2x^2+1\right)\right)=\frac{1}{2x^2+1}(4x) \quad \textbf{[M1]}$$

$$=\frac{4x}{2x^2+1} \quad \textbf{[A1]}$$

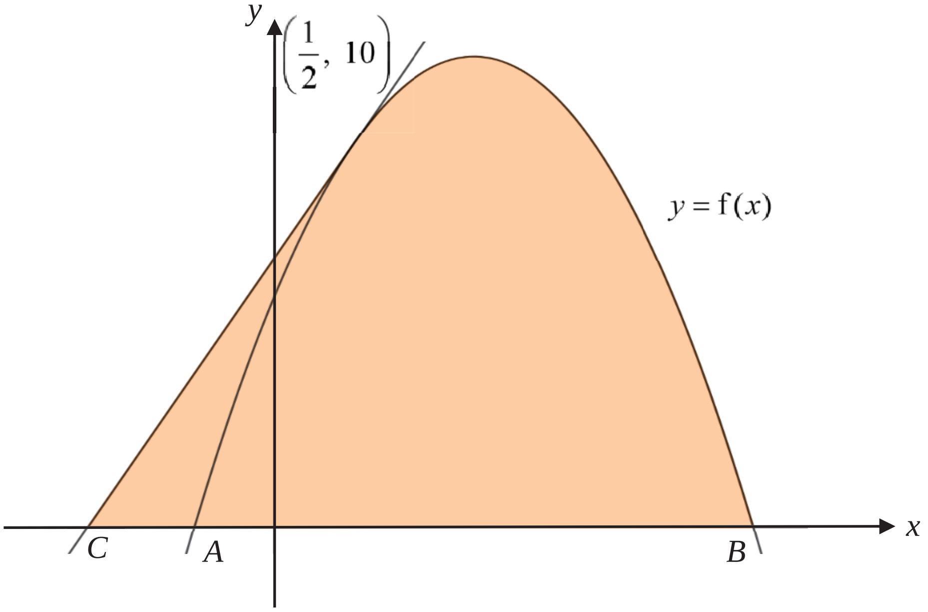
(iii) Using the results from part **(i)** and **(ii)**, find $\displaystyle\int\frac{2x^3-20x^2-17x-10}{\left(x^2-4\right)\left(2x^2+1\right)}\,dx$. [3]

$$\int\frac{2x^3-20x^2-17x-10}{\left(x^2-4\right)\left(2x^2+1\right)}\,dx=\int-\frac{3}{x-2}+\frac{2}{x+2}+\frac{4x}{2x^2+1}\,dx \quad \textbf{[M1]}$$

$$=-3\ln(x-2)+2\ln(x+2)+\ln(2x^2+1)+c$$

[**A2** for all correct, A1 for 2 correct terms, award mark if ecf from part (i) or (ii)]

7



The diagram shows the curve $y = f(x)$ which intersects the x-axis at points A and B. The tangent to the curve at the point $\left(\frac{1}{2}, 10\right)$ intersects the x-axis at point C. It is given that $\frac{dy}{dx} = -8x + 10$.

(i) Show that B is the point $(3, 0)$ and find the coordinates of C. [5]

$\frac{dy}{dx} = -8x + 10$

$y = \int(-8x+10)dx$ **[M1]**

$= -4x^2 + 10x + c$

At the point $(\frac{1}{2}, 10)$,

$10 = -4(\frac{1}{2})^2 + 10(\frac{1}{2}) + c$

$c = 6$

$y == -4x^2 + 10x + 6$ **[M1]**

$y = -2(2x^2 - 5x - 3)$

$= -2(2x+1)(x-3)$

When $y = 0$, $x = -\frac{1}{2}$ or $x = 3$

Coordinates of $B = (3, 0)$ (shown) **[A1]**

At the point $(\frac{1}{2},\ 10)$, $\frac{dy}{dx} = 6$

Equation of tangent line:

$y - 10 = 6(x - \frac{1}{2})$ **[M1]**

$y = 6x + 7$

When $y = 0$, $x = -\frac{7}{6}$

Coordinates of $C = (-\frac{7}{6},\ 0)$ **[A1]**

(ii) Find the area of the shaded region. [5]

Area of shaded region $= \frac{1}{2}(10)(\frac{1}{2} + \frac{7}{6}) + \int_{\frac{1}{2}}^{3} \left(-4x^2 + 10x + 6\right) dx$ **[M1 + M1]**

$= 8\frac{1}{3} + \left[-\frac{4x^3}{3} + 5x^2 + 6x \right]_{\frac{1}{2}}^{3}$ **[M1]**

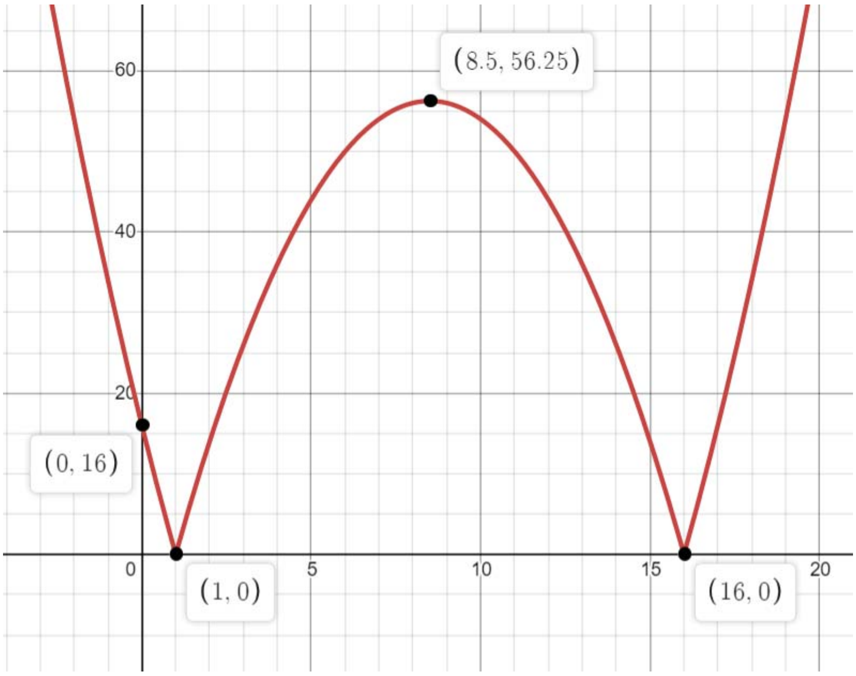
$= 8\frac{1}{3} + \left[-\frac{4(3)^3}{3} + 5(3)^2 + 6(3) \right] - \left[-\frac{4(\frac{1}{2})^3}{3} + 5(\frac{1}{2})^2 + 6(\frac{1}{2}) \right]$

[M1]

$= 8\frac{1}{3} + 27 - 4\frac{1}{12}$

$= 31.25$ units^2 **[A1]**

8 **(i)** Sketch the graph of $y = \left| x^2 - 17x + 16 \right|$. [3]



Shape of graph **[C1]**

Turning point **[C1]**

x and *y*-intercept(s) **[C1]**

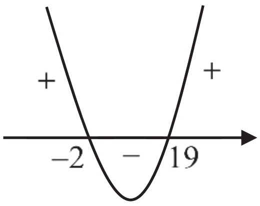
(ii) Find the range of values of x for which $\left|x^2 - 17x + 16\right| > 54$. [5]

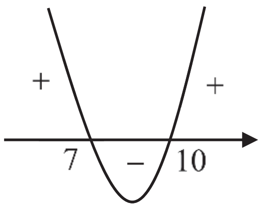
$\left|x^2 - 17x + 16\right| > 54$

$x^2 - 17x + 16 > 54$ or $-x^2 + 17x - 16 > 54$ **[M1]**

$x^2 - 17x - 38 > 0$ or $x^2 - 17x + 70 < 0$

$(x - 19)(x + 2) > 0$ **[M1]** or $(x - 7)(x - 10) < 0$ **[M1]**





$x < -2$ or $x > 19$ **[A1]** $7 < x < 10$ **[A1]**

The range of values of x is $x < -2$, $7 < x < 10$ and $x > 19$.

Alternative:

For $\left|x^2 - 17x + 16\right| = 54$,

$x^2 - 17x + 16 = 54$ or $x^2 - 17x + 16 = -54$ **[M1]**

$x^2 - 17x - 38 = 0$ or $x^2 - 17x + 70 = 0$

$(x - 19)(x + 2) = 0$ or $(x - 7)(x - 10) = 0$

$x = 19$ or $x = -2$ **[M1]** or $x = 7$ or $x = 10$ **[M1]**

For $\left|x^2 - 17x + 16\right| > 54$,

the range of values of x is $x < -2$, $7 < x < 10$ and $x > 19$.

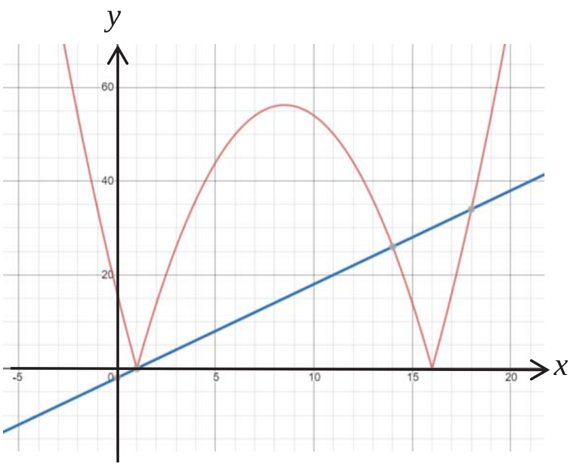
[**A1** for $x < -2$ or $x > 19$ and **A1** for $7 < x < 10$]

(iii) Given that $|x^2 - 17x + 16| = k$ has more than 2 distinct solutions, state the range of values of k. [2]

The range of values of k is $0 < k \leq 56\frac{1}{4}$. **[B1 + B1]**

(iv) Determine the number of solutions of the equation $|x^2 - 17x + 16| = 2x - 2$. Justify your answer. [2]

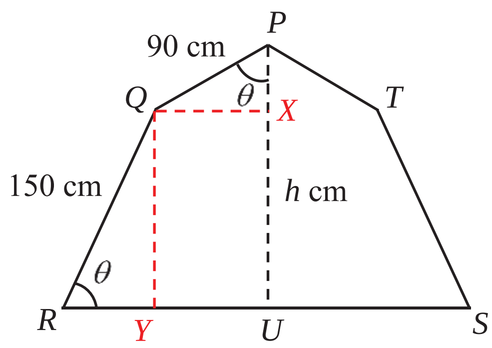
For $y = 2x - 2$, when $y = 0$, $x = 1$.



[M1 for sketching $y = 2x - 2$, clearly showing the x-intercept**]**

Since the line $y = 2x - 2$ intersects the curve at (1, 0) and the gradient is 2, the number of solutions is 3. **[A1]**

9 The diagram shows the side view *PQRST* of a tent. The tent rests with *RS* on horizontal ground. *PQRST* is symmetrical about the vertical *PU*, where *U* is the midpoint of *RS*. Angle *QPU* = angle *QRU* = θ radians and the lengths of *PQ* and *QR* are 90 cm and 150 cm respectively. The vertical height of *P* from the ground is *h* cm.



(i) Explain clearly why $h = 90\cos\theta + 150\sin\theta$. [2]

Let the foot of the perpendicular from *Q* to *PU* and *RU* be *X* and *Y* respectively.

$PX = 90\cos\theta$
$QY = 150\sin\theta$ } **[M1** for both**]**

$h = PX + QY$

$= 90\cos\theta + 150\sin\theta$ **[A1]**

(ii) Express *h* in the form $R\cos(\theta - \alpha)$, where $R > 0$ and α is an acute angle. [4]

$R\cos\alpha = 90$

$R\sin\alpha = 150$ **[M1]**

$\tan\alpha = \dfrac{150}{90}$ **[M1]**

$\alpha = 1.0304$

$R = \sqrt{90^2 + 150^2}$

$= 30\sqrt{34}$ **[M1]**

$h = 30\sqrt{34}\cos(\theta - 1.03)$ **[A1 -** accept if $R = 175$ (3s.f.)**]]**

(iii) Find the greatest possible value of *h* and the value of θ at which this occurs. [3]

Greatest value of *h* occurs when $\cos(\theta - 1.03) = 1$ **[M1]**

Greatest value of $h = 30\sqrt{34}$ **[A1** – Accept 175**]**

when $\theta = 1.03$. **[A1]**

(iv) Find the values of θ when h = 160. [3]

$30\sqrt{34}\cos(\theta - 1.0304) = 160$

$\cos(\theta - 1.0304) = \dfrac{160}{30\sqrt{34}}$ **[M1]**

Basic angle = 0.41613

$\theta - 1.0304 = -0.41613$ or 0.41613

$\theta = 0.614$ or 1.45 (to 3s.f.) **[A1, A1]**

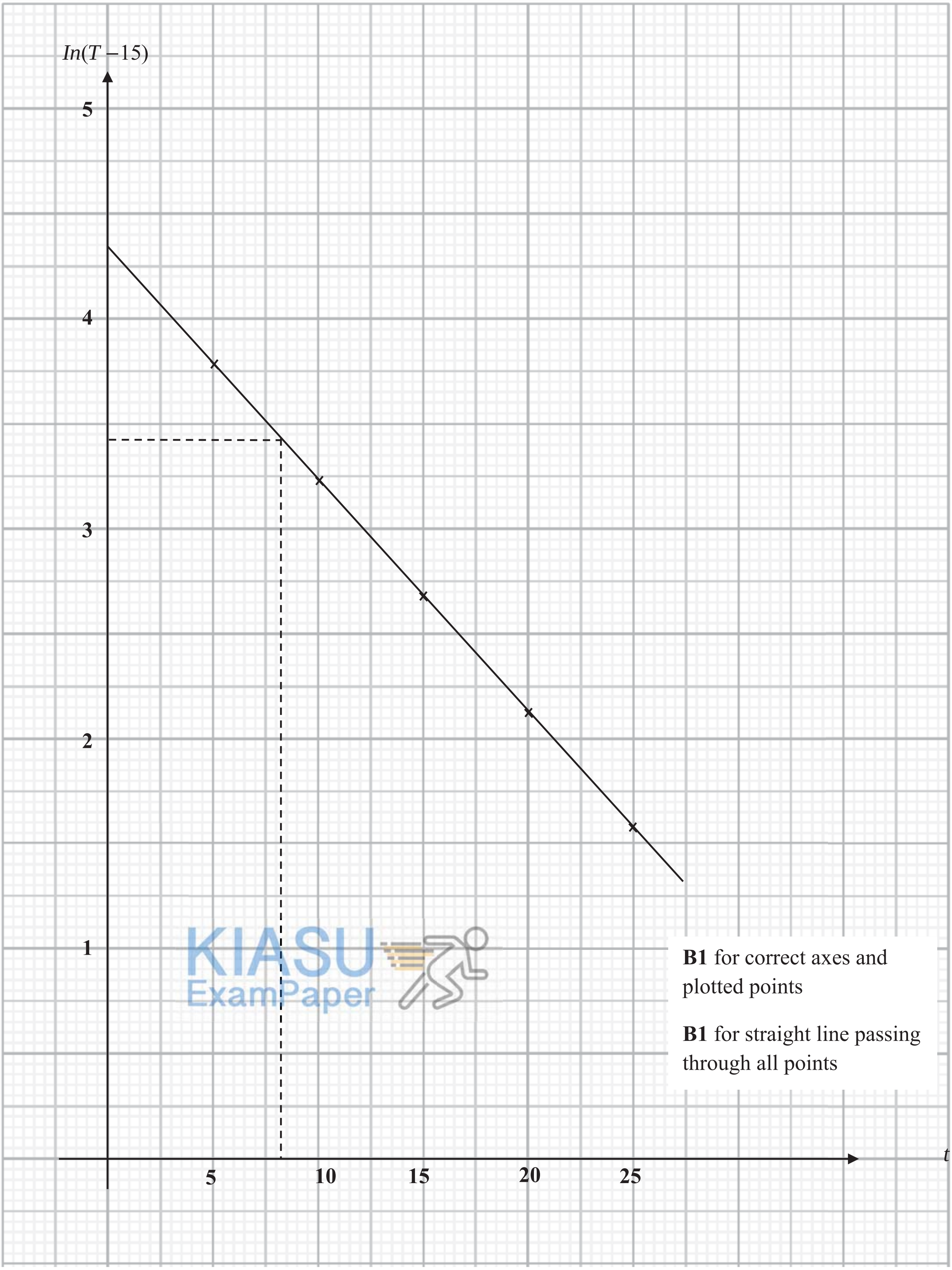
10 A bowl of liquid is heated to a high temperature. It subsequently cools in such a way that its temperature, T°C, is given by $T = 15 + Ae^{-kt}$, where t minutes is the time of cooling and A and k are constants. The table below shows corresponding values of t and T.

t	5	10	15	20	25
T	58.8	40.3	29.6	23.4	19.9

(i) Draw the graph of $\ln(T - 15)$ against t. [3]

t	5	10	15	20	25
T	58.8	40.3	29.6	23.4	19.9
$\ln(T - 15)$	3.78	3.23	2.68	2.13	1.59

[B1 for table**]**



B1 for correct axes and plotted points

B1 for straight line passing through all points

(ii) Use the graph to estimate the value of each of the constants A and k. [5]

$T = 15 + Ae^{-kt}$

$\ln(T - 15) = -kt + \ln A$ **[M1]**

gradient = $-k$, vertical-intercept = $\ln A$ **[M1]**

$\ln A = 4.3$

$A = e^{4.325}$

$= 75.6$ (accept 73.7 to 77.5) **[A1]**

$-k = -0.11$ **[M1** for finding gradient**]**

$k = 0.11$ (accept 0.105 to 0.115) **[A1]**

(iii) State the initial temperature of the liquid. [1]

$T = 15 + 75.6e^{-0.11(0)}$

$= 90.6$

The initial temperature is $90.6°\text{C}$. (accept 88.7 to 92.5) **[B1]**

(iv) Use the graph to estimate the time taken for the temperature of the liquid to drop to half of its original temperature. [2]

When the temperature of the liquid is half of its original temperature,

$\ln(T - 15) = \ln\left(\frac{90.6}{2} - 15\right)$

$= 3.41$ **[M1]**

Time taken = 8.25 minutes (accept 8 to 8.5 minutes) **[A1]**