Answer Key

1. $1 < k < 5$

3. (ii) $x = -1,\ \dfrac{1}{2}$ or 3

4. (i) $\left(x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$

5. (i) $E(2, 2)$

   (ii) $B\left(3\dfrac{1}{3},\ 1\dfrac{1}{3}\right)$

   (iii) $D\left(3\dfrac{1}{3},\ 4\dfrac{2}{3}\right)$

   (iv) $13\dfrac{8}{9}$ units^2

6. $\dfrac{dy}{dx} = \dfrac{9}{(x-3)(x+6)}$, y is an increasing function.

7. (ii) $a = 2.40$ (allow between 2.29 to 2.51), $b = 2.01$ (allow 1.95 to 2.1)

   (iii) Draw $\lg y = 2 - \dfrac{x}{3}$

   (iv) $k = 4.63$ [$\lg y$ -intercept + (14)(gradient of line)]

8. Maximum $A = 25200$

9. (a) $x = 1$

   (b) $12 - 5\sqrt{3}$

10. (ii) $\dfrac{1 - 2\sqrt{3}}{2\sqrt{5}}$

11. $\sin(2x+1) - 2x\cos(2x+1) + c$

12. (i) $\dfrac{dy}{dx} = 1 - \dfrac{1}{4}x$, $y = \dfrac{3}{4}x$

    (ii) $p = -\dfrac{1}{24}$

Answer Key

1(i) -22.5 units/s (ii) $k=4$

2(a) $x=3$ (b) $y=x^4$

(c) $(e^{2x})^2+3e^{2x}+7=0$.

Since discriminant < 0, there are no real solutions for the equation. (Or equivalent methods)

3(i) $-\dfrac{11}{4}$ (iii) $x^2-\dfrac{63}{20}x+\dfrac{5}{2}=0$

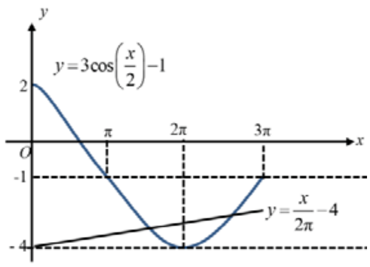
4(ii) $x=\dfrac{\pi}{8}$ or $\dfrac{3\pi}{8}$

5(i) $1+7x+21x^2+35x^3+\ldots$ (ii) $T_{r+1}=\dbinom{9}{r}x^{2(9-r)}\left(-\dfrac{2}{x^3}\right)^r$ (or equivalent)

(iii) $18-5r$ (iv) -637

6(i) amplitude = 3; period = 4π (ii) $k=-1$

(iii)



(iv) number of solutions = 2

7(ii) $L=60+6\sqrt{58}\sin(\theta-23.2°)$ (iii) $\theta=67.6°$

8(i) $a=2,\ b=1$ and $c=-3$ (ii) $\dfrac{x-3}{2x^2-x}=\dfrac{3}{x}-\dfrac{5}{2x-1}$ (iii) $4x+6\ln x-5\ln(2x-1)+c$

9(i) acceleration = -0.439 m/s^2 (3 s.f.) (iii) distance = 3.07 m (3 s.f.)

10(iii) 12 units^2

11(i) $(x+9)^2+(y-8)^2=25$ (ii) C is (10, 0) (iii) 10 units

Name:	Class:	Class Register Number:

中正中学

CHUNG CHENG HIGH SCHOOL (MAIN)

PRELIMINARY EXAMINATION 2020
SECONDARY 4

ADDITIONAL MATHEMATICS (Solutions) 4047/01

Paper 1 **Tuesday 15 September 2020**

2 hours

Candidates answer on the Question Paper.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staple, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 80.

For Candidate's Use	For Examiner's Use
Question Number	Marks Obtained
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
Total Marks	

This document consists of **20** printed pages.

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 Find the range of values of the constant k for which the curve $y = x^2 + 3x + 4$ lies entirely above the line $y = kx + 3$. [4]

$x^2 + 3x + 4 > kx + 3$

$x^2 + 3x - kx + 1 > 0$

$x^2 + (3-k)x + 1 > 0$

(The resultant quadratic equation is always positive, thus there is no real roots)

OR

$y = x^2 + 3x + 4$ --------------- (1)

$y = kx + 3$ ---------- (2)

Sub. (1) into (2)

$kx + 3 = x^2 + 3x + 4$

$x^2 + 3x - kx + 1 = 0$

$x^2 + (3-k)x + 1 = 0$

(The resultant quadratic equation has no real roots since the line and curve do not intersect)

Discriminant < 0 (Use discriminant to formula an inequality in k only)

$(3-k)^2 - 4(1)(1) < 0$

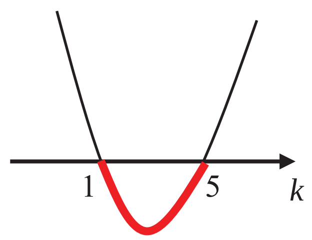
$(3-k)^2 - 2^2 < 0$

$(3-k-2)(3-k+2) < 0$ (Factorisation by 'difference of 2 squares'. Alternatively, expand and perform 'cross-factorisation)

$(1-k)(5-k) < 0$

$1 < k < 5$

(This solution is obtained using the quadratic sketched below)



[Turn over

2 Sketch the graph of $y = |x^2 - 1|$, showing the **coordinate(s)** where the graph meets the x-axis and the turning point of the graph. [3]

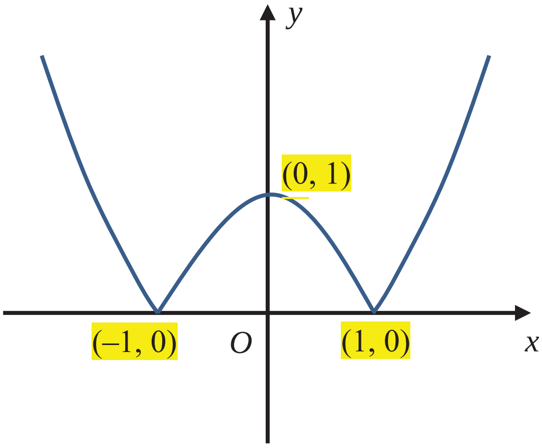
When $y = 0$,

$|x^2 - 1| = 0$

$x^2 - 1 = 0$

$x^2 = 1$

$x = -1$ or 1



3 A polynomial $f(x) = 2x^3 - 5x^2 + ax + b$, where a and b are constants, leaves a remainder of -4 when divided by $x - 1$. The graph of $y = f(x)$ has a stationary point at $x = 2$.

(i) Show that $a = -4$ and $b = 3$. [5]

$f(x) = 2x^3 - 5x^2 + ax + b$

$f'(x) = 6x^2 - 10x + a$

$f'(2) = 0$ (At stationary point x = 2, the gradient of curve = 0)

$6(2)^2 - 10(2) + a = 0$

$24 - 20 + a = 0$

$a = -4$

$f(x) = 2x^3 - 5x^2 - 4x + b$

$f(1) = -4$ (When divided by $x - 1$, the remainder is –4)

$2(1)^3 - 5(1)^2 - 4(1) + b = -4$

$2 - 5 - 4 + b = -4$

$b = 3$

(ii) Show that $x+1$ is a factor of $\mathrm{f}(x)$ and solve the equation $\mathrm{f}(x)=0$. [4]

$$\mathrm{f}(x)=2x^3-5x^2-4x+3$$
$$\mathrm{f}(-1)=2(-1)^3-5(-1)^2-4(-1)+3$$
$$=0$$

Since the remainder is 0, by factor theorem $x+1$ is a factor.

$$\begin{array}{r|l} & 2x^2-7x+3 \\ \hline x+1 & 2x^3-5x^2-4x+3 \\ & -(2x^3+2x^2) \\ \hline & \quad -7x^2-4x \\ & \quad -(-7x^2-7x) \\ \hline & \qquad 3x+3 \\ & \qquad -(3x+3) \\ \hline & \qquad\quad 0 \end{array}$$

$$(x+1)(2x^2-7x+3)=0$$

$$(x+1)(2x-1)(x-3)=0$$

$$x=-1,\ \frac{1}{2}\ \text{or}\ 3$$

4 **(i)** By completing the square, express $x^2 - x + 1$ in the form $(x-p)^2 + q$ where p and q are constants. [2]

$$x^2 - x + 1 = x^2 - x + \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 + 1$$
$$= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

(ii) Show that the curve $y = x^2 - 2px + p - 1$ will always cut the x-axis at two distinct points for all real values of p. [3]

$$\text{Discriminant} = (-2p)^2 - 4(1)(p-1)$$
$$= 4p^2 - 4p + 4$$
$$= 4\left(p^2 - p + 1\right)$$

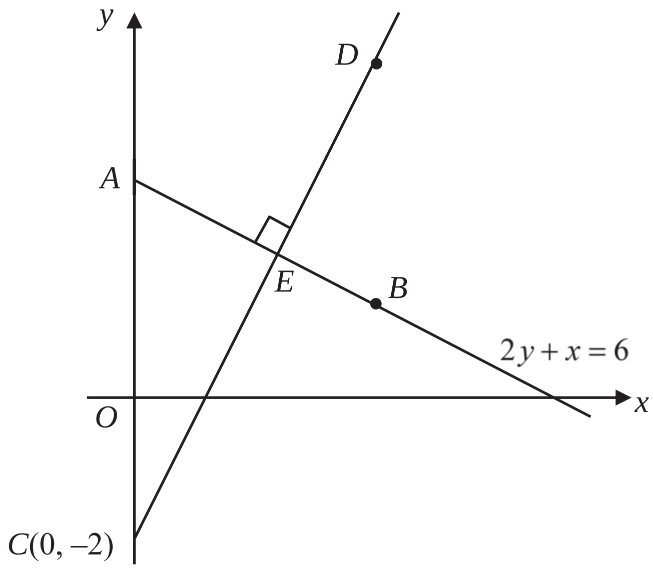
[Apply the result from (i)]

$$= 4\left[\left(p - \frac{1}{2}\right)^2 + \frac{3}{4}\right]$$
$$= 4\left(p - \frac{1}{2}\right)^2 + 3$$

Since $\left(p - \frac{1}{2}\right)^2$ is never negative for all real values of p, thus the discriminant $4\left(p - \frac{1}{2}\right)^2 + 3$ will always be positive. Hence the curve will always cut the x-axis at two distinct points.

[Turn over

5 In the diagram, the line $2y + x = 6$ cuts the y-axis at point A and passes through point B. The line CD cuts the y-axis at $(0, -2)$ and intersects line AB at point E. The two lines AB and CD are perpendicular to each other.



(i) Find the coordinates of E. [4]

$2y + x = 6$

$2y = -x + 6$

$y = -\frac{1}{2}x + 3$ ------- (1)

Gradient of line AB is $-\frac{1}{2}$.

Gradient of CD is 2. (Apply $m_1 m_2 = -1$)

Equation of CD is

$y = 2x - 2$ -------- (2)

Sub. (1) into (2),

$2(2x - 2) + x = 6$

$4x - 4 + x = 6$

$5x = 10$

$x = 2$ sub. into (1)

$y = 2$

$E(2, 2)$

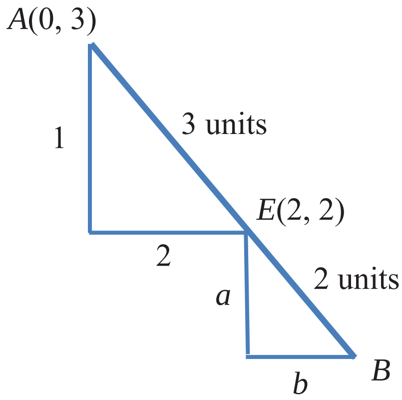
(ii) The ratio of $AE : AB$ is 3 : 5. Find the coordinates of B. [4]

$2y + x = 6$

When $x = 0$, $2y = 6$ **B1**

$y = 3$

$A(0,3)$



Using similar triangles,

$$\frac{a}{1} = \frac{2}{3}$$

$$a = \frac{2}{3}$$ (Application of similar triangles)

* $$\frac{b}{2} = \frac{2}{3}$$ (Application of similar triangles)

$$b = \frac{4}{3}$$

$$\therefore B\left(2 + \frac{4}{3},\ 2 - \frac{2}{3}\right) = B\left(3\frac{1}{3},\ 1\frac{1}{3}\right)$$

*** Alternative Method**

At point B, $x = 2 + \frac{4}{3} = 3\frac{1}{3}$ sub. into eq. of AB

$$2y + 3\frac{1}{3} = 6$$

$$y = 1\frac{1}{3}$$

$$\therefore B\left(3\frac{1}{3}, 1\frac{1}{3}\right)$$

Alternative Method

$5AE = 3AB$

$$5\sqrt{(2-0)^2 + (2-3)^2} = 3\sqrt{(x-0)^2 + (y-3)^2}$$

$$25(4+1) = 9\left[x^2 + (y-3)^2\right]$$

$$\frac{125}{9} = x^2 + (y-3)^2 \quad \text{----------- (1)}$$

$2y + x = 6$

$x = 6 - 2y$ -------- (2)

Sub. (2) into (1)

$$\frac{125}{9} = (6-2y)^2 + (y-3)^2$$

$$\frac{125}{9} = 4(y-3)^2 + (y-3)^2$$

$$5(y-3)^2 = \frac{125}{9}$$

$$(y-3)^2 = \frac{25}{9}$$

$$y - 3 = -\frac{5}{3} \text{ or } \frac{5}{3}$$

$$y = 1\frac{1}{3} \text{ or } 4\frac{2}{3} \text{ (rej.)}$$

Sub. $y = 1\frac{1}{3}$ into (2),

$$x = 3\frac{1}{3}$$

$$\therefore B\left(3\frac{1}{3}, 1\frac{1}{3}\right)$$

Given that BD is parallel is the y-axis.

(iii) Find the coordinates of D. [2]

Sub. $x = 3\frac{1}{3}$ into $y = 2x - 2$

$$y = 2\left(3\frac{1}{3}\right) - 2$$

$$= 4\frac{2}{3}$$

$$\therefore D\left(3\frac{1}{3},\ 4\frac{2}{3}\right)$$

(iv) Find the area of quadrilateral $ACBD$. [2]

$$\text{Area of quadrilateral } ACBD = \frac{1}{2}\begin{vmatrix} 0 & 0 & 3\frac{1}{3} & 3\frac{1}{3} & 0 \\ 3 & -2 & 1\frac{1}{3} & 4\frac{2}{3} & 3 \end{vmatrix}$$

$$= \frac{1}{2}\left[\left(\frac{140}{9} + 10\right) - \left(\frac{40}{9} - \frac{20}{3}\right)\right]$$

$$= 13\frac{8}{9} \text{ units}^2$$

6 The equation of a curve is $y=\ln\left(\frac{x-3}{x+6}\right)$ for $x>3$. Determine whether y is an increasing or decreasing function. [5]

$$y=\ln\left(\frac{x-3}{x+6}\right)$$
$$=\ln(x-3)-\ln(x+6)$$

(Simplifying using the laws of logarithms will reduce the complexity of the technique of differentiation.)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x-3}-\frac{1}{x+6}$$

[Note: $\frac{\mathrm{d}}{\mathrm{d}x}(\ln u)=\frac{u'}{u}$]

$$=\frac{x+6-(x-3)}{(x-3)(x+6)}$$
$$=\frac{9}{(x-3)(x+6)}$$

Alternative Method

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{(x+6)(1)-(x-3)(1)}{(x+6)^2}}{\left(\frac{x-3}{x+6}\right)}$$

This is NOT recommended!

$$=\frac{x+6-x+3}{(x+6)^2}\times\frac{x+6}{x-3}$$

$$=\frac{9}{(x-3)(x+6)}$$

For $x>3$, 9 is a positive constant, $x-3$ and $x+6$ are both positive.

Since $\frac{\mathrm{d}y}{\mathrm{d}x}>0$. Hence y is an increasing function.

7 Variables x and y are related by the equation $y = ab^x$ where a and b are constants.

The table below shows values of x and y.

x	1	2	3	4	5	6
y	4.8	9.6	19.2	38.4	76.8	153.6

(i) Draw the graph of $\lg y$ against x, using a scale of 2 cm for 1 unit on the x-axis and 1 cm for 0.1 unit on the $\lg y$-axis. [3]

x	1	2	3	4	5	6
$\lg y$	0.681	0.982	1.28	1.58	1.89	2.19

(ii) Use the graph to estimate the value of a and of b. [3]

$y = ab^x$

$\lg y = \lg ab^x$

$\lg y = \lg a + \lg b^x$

$\lg y = \lg a + x \lg b$

From the graph,

$\lg b = \text{gradient}$

$= \dfrac{2.20 - 0.38}{6 - 0}$

$= \dfrac{91}{300}$

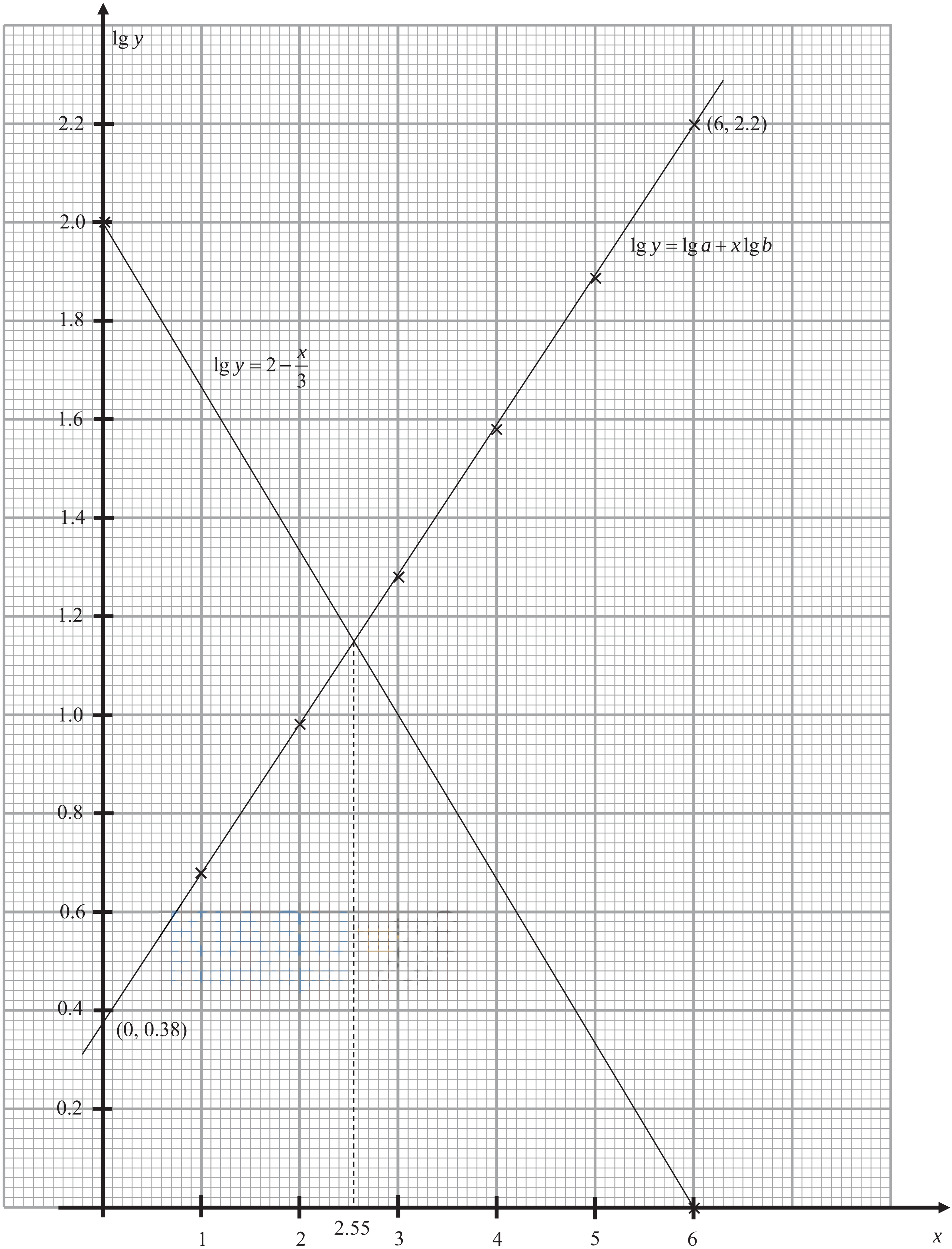
$b = 10^{\frac{91}{300}}$

$= 2.01$ (3 s.f)

$\lg a = 0.38$

$a = 10^{0.38}$

$= 2.40$ (3 s.f)


lg y
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
(6, 2.2)
lg y = lg a + x lg b
lg y = 2 − x/3
(0, 0.38)
1
2
2.55
3
4
5
6
x

[Turn over

(iii) By adding a suitable straight line to your graph in **part (i)**, estimate the solution to the equation $ab^x = 10^{2-\frac{x}{3}}$. [2]

$ab^x = 10^{2-\frac{x}{3}}$

$\lg ab^x = \lg 10^{2-\frac{x}{3}}$

$\lg y = \lg 10^{2-\frac{x}{3}}$

$= \left(2 - \frac{x}{3}\right)\lg 10$

Draw $\lg y = 2 - \frac{x}{3}$ (Draw $Y = 2 - \frac{X}{3}$ which is a straight line)

From the graph, $x = 2.55$

(iv) The point $(14, k)$ lies on the graph of $\lg y$ against x, using the values of a and b that were found in part (ii), find the value of k. [1]

$\lg y = 0.38 + \frac{91}{300}x$

$k = 0.38 + \frac{91}{300}(14)$

$= 4.63$ (3 s.f)

8 The surface area of an object, A cm^2, is given by $A = 300x - \left(\dfrac{\pi+4}{8}\right)x^2$. Given that x cm can vary, find the stationary value of A and determine whether it is a maximum or minimum. [5]

$$A = 300x - \left(\frac{\pi+4}{8}\right)x^2$$

$$\frac{dA}{dx} = 300 - \frac{2(\pi+4)}{8}x$$

$$= 300 - \frac{\pi+4}{4}x$$

At stationary A, $\dfrac{dA}{dx} = 0$

$$300 - \frac{\pi+4}{4}x = 0$$

$$x = \frac{1200}{\pi+4}$$

$$= 168.0297...$$

$$\frac{d^2A}{dx^2} = \frac{-\pi-4}{4} < 0$$

Thus when $x = 168.0297...$,

Maximum $A = 300(168.0297...) - \dfrac{\pi+4}{8}(168.0297...)^2$

$$= 25204.461...$$

$$= 25200 \text{ (3 s.f)}$$

9 **(a)** Solve the equation $\sqrt{8-4x}+\sqrt{2-x}=3x$. [4]

$$\sqrt{8-4x}+\sqrt{2-x}=3x$$
$$2\sqrt{2-x}+\sqrt{2-x}=3x$$

$$3\sqrt{2-x}=3x$$
$$\sqrt{2-x}=x$$
$$2-x=x^2$$

$$x^2+x-2=0$$
$$(x+2)(x-1)=0$$

Alternative Method

$$\left(\sqrt{8-4x}+\sqrt{2-x}\right)^2=9x^2$$
$$8-4x+2\sqrt{8-4x}\sqrt{2-x}+2-x=9x^2$$
$$10-5x+2\sqrt{4(2-x)}\sqrt{2-x}=9x^2$$
$$10-5x+4(2-x)=9x^2$$
$$10-5x+8-4x=9x^2$$
$$9x^2-9x+18=0$$
$$x^2-x+2=0$$

$$(x+2)(x-1)=0$$
$$x=1 \text{ or } -2 \text{ (rej.)}$$

(b) A cuboid of volume $\left(54-11\sqrt{3}\right)$ cm^3 has a base area $\left(7+2\sqrt{3}\right)$ cm^2 and height h cm.

Without using a calculator, obtain an expression for h in the form $a+b\sqrt{3}$, where a and b are integers. [4]

$$h=\frac{54-11\sqrt{3}}{7+2\sqrt{3}} \quad \left(\text{height}=\frac{\text{Volume}}{\text{base area}}\right)$$
$$=\frac{54-11\sqrt{3}}{7+2\sqrt{3}}\times\frac{7-2\sqrt{3}}{7-2\sqrt{3}} \quad \text{(Multiplication by conjugate surds of the denominator)}$$
$$=\frac{378-108\sqrt{3}-77\sqrt{3}+(22)(3)}{(7)^2-(2\sqrt{3})^2}$$
$$=\frac{444-185\sqrt{3}}{49-12}$$
$$=\frac{444-185\sqrt{3}}{37}$$
$$=12-5\sqrt{3}$$

10 Given that $6\sin^2 A - 4\cos^2 A = 5\sin 2A$ where $0 \le A \le 90°$.

(i) Show that $\tan A = 2$. [3]

$$6\sin^2 A - 4\cos^2 A = 5\sin 2A$$
$$6\sin^2 A - 4\cos^2 A = 5\left(2\sin A\cos A\right)$$
$$3\sin^2 A - 5\sin A\cos A - 2\cos^2 A = 0$$
$$\left(3\sin A + \cos A\right)\left(\sin A - 2\cos A\right) = 0$$
$$3\sin A = -\cos A \quad \text{or} \quad \sin A = 2\cos A$$
$$\frac{\sin A}{\cos A} = -\frac{1}{3} \quad \text{or} \quad \frac{\sin A}{\cos A} = 2$$
$$\tan A = -\frac{1}{3} \text{ (rej. } \because A \text{ is acute)} \quad \text{or} \quad \tan A = 2 \text{ (shown)}$$

Alternative Method

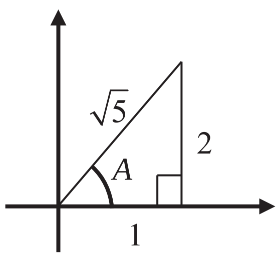
$$6\sin^2 A - 4\cos^2 A = 5\sin 2A$$
$$6\sin^2 A - 4\cos^2 A = 5\left(2\sin A\cos A\right)$$
$$\frac{6\sin^2 A}{\cos^2 A} - \frac{4\cos^2 A}{\cos^2 A} = \frac{10\sin A\cos A}{\cos^2 A}$$
$$6\tan^2 A - 4 = 10\tan A$$
$$6\tan^2 A - 10\tan A - 4 = 0$$

$$3\tan^2 A - 5\tan A - 2 = 0$$
$$\left(3\tan A + 1\right)\left(\tan A - 2\right) = 0$$
$$\tan A = 2 \text{ (shown) or } \tan A = -\frac{1}{3} \text{ (rej.)}$$

(ii) **Hence,** find value of $\cos\left(60^\circ + A\right)$, leaving your answer in the form $\dfrac{a + b\sqrt{3}}{2\sqrt{5}}$. [3]

$$\cos\left(60° + A\right) = \cos 60°\cos A - \sin 60°\sin A$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{5}}\right) - \frac{\sqrt{3}}{2}\left(\frac{2}{\sqrt{5}}\right)$$
$$= \frac{1 - 2\sqrt{3}}{2\sqrt{5}}$$

(Draw a diagram to find the other trigo. Ratios in terms of A)



(iii) Without finding the value of A, explain whether $60° + A$ is acute or obtuse. [1]

Since $1 - 2\sqrt{3}$ is negative, from (ii), $\cos\left(60° + A\right)$ is a negative ratio. So $60° + A$ lies either in the 2nd or 4th quadrant. Since both $60°$ and A are both acute, the addition of 2 acute angles cannot exceed $180°$. Thus $60° + A$ lies in the 2nd quadrant and is obtuse.

11 Differentiate $x\cos(2x+1)$ with respect to x. Hence, determine $\int 4x\sin(2x+1)\ \mathrm{d}x$. [6]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x\cos(2x+1)\right] = x\left[-2\sin(2x+1)\right] + \left[\cos(2x+1)\right](1)$$

$$= -2x\sin(2x+1) + \cos(2x+1)$$

$$\int\left[\cos(2x+1) - 2x\sin(2x+1)\right]\ \mathrm{d}x = x\cos(2x+1) + c \quad \text{where } c \text{ is a constant}$$

$$\int\cos(2x+1)\ \mathrm{d}x - \int 2x\sin(2x+1)\ \mathrm{d}x = x\cos(2x+1) + c$$

$$\int 2x\sin(2x+1)\ \mathrm{d}x = \int\cos(2x+1)\ \mathrm{d}x - x\cos(2x+1) + c_1 \quad \text{where } c_1 \text{ is a constant}$$

$$\int 4x\sin(2x+1)\ \mathrm{d}x = 2\int\cos(2x+1)\ \mathrm{d}x - 2x\cos(2x+1) + c_2$$

$$= \frac{2\sin(2x+1)}{2} - 2x\cos(2x+1) + c_3 \quad \text{where } c_3 \text{ is a constant}$$

$$= \sin(2x+1) - 2x\cos(2x+1) + c_3$$

12 A curve has the equation $y=-2\left(1-\frac{1}{4}x\right)^2+p$.

(i) Find $\frac{dy}{dx}$. Hence, find the equation of the line that is parallel to the tangent of the curve at $x=1$ and passes through the origin. [3]

$$y=-2\left(1-\frac{1}{4}x\right)^2+p$$

$$\frac{dy}{dx}=-4\left(1-\frac{1}{4}x\right)\left(-\frac{1}{4}\right)$$

$$=1-\frac{1}{4}x$$

When $x=1$, gradient of tangent $=\frac{3}{4}$

Equation of line is $y=\frac{3}{4}x$ (Since the line passes though the origin, the y-intercept is 0)

The normal to the curve at $x=1$ passes through the point $\left(-1,\ 1\frac{1}{2}\right)$.

(ii) Find the value of the constant p, [4]

Gradient of normal $=-\frac{4}{3}$

At $x=1$, $y=-2\left(1-\frac{1}{4}\right)^2+p$

$$=-\frac{9}{8}+p$$

$$\frac{-\frac{9}{8}+p-1\frac{1}{2}}{1-(-1)}=-\frac{4}{3}$$

$$-\frac{21}{8}+p=-\frac{8}{3}$$

$$p=-\frac{1}{24}$$

(It is not essential to find equation of normal, you can formulate the gradient of normal using $\frac{y_2-y_1}{x_2-x_1}$ and equate to the gradient that is found through differentiation)

Name:	Class:	Class Register Number:

中正中学

CHUNG CHENG HIGH SCHOOL (MAIN)

PRELIMINARY EXAMINATION 2020

SECONDARY 4

ADDITIONAL MATHEMATICS (Solutions)

4047/02

Thursday 17 September 2020

Paper 2

2 hours 30 minutes

Candidates answer on the Question Paper.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 100.

For Candidate's Use	**For Examiner's Use**
Question Number	Marks Obtained
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total Marks	

This document consists of **19** printed pages and **1** blank page.

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 It is given that $y = \dfrac{2}{x-1}$ and y is increasing at a constant rate of 5 units per second.

(i) Find the rate of change of x when $x = 4$. [3]

(i)

$$y = \frac{2}{x-1}$$

$$\frac{dy}{dx} = \frac{0-2}{(x-1)^2}$$

$$= -\frac{2}{(x-1)^2}$$

When $x = 4$,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$ (Write down the connected rates of change equation)

$$5 = \left(-\frac{2}{3^2}\right) \times \frac{dx}{dt}$$

$$\frac{dx}{dt} = 5 \div \left(-\frac{2}{9}\right)$$

$$= -22.5 \text{ units/s}$$

Rate of change of x is -22.5 units/s.

OR

Comments

- A common error when using quotient rule to find $\dfrac{dy}{dx}$ is forgetting to differentiate the constant 2.
- "Rate of change of x" refers to $\dfrac{dx}{dt}$, not $\dfrac{dx}{dy}$ which means rate of change of x **with respect to *y***.
- In this context, we are given $\dfrac{dy}{dt}$ and we are interested to find out how x would change accordingly ($\dfrac{dx}{dt}$).

(ii) Find the value of the constant k for which $(x-1)^3\left(\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx}\right) = kx$. [4]

(ii)

$$\frac{d^2y}{dx^2} = \frac{0-(-2)\times 2(x-1)}{(x-1)^2}$$

$$= \frac{4(x-1)}{(x-1)^4}$$

$$= \frac{4}{(x-1)^3}$$

$$(x-1)^3\left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\right) = (x-1)^3\left[\frac{4}{(x-1)^3} - 2\left(\frac{-2}{(x-1)^2}\right)\right]$$

$$= 4 + 4(x-1)$$

$$= 4x$$

$\therefore k = 4$

Comments

- Quite a number of students simplified this question by letting $x = 4$ and arriving at the same answer. ✗
- It should be noted, however, there is no indication that $x = 4$ is a solution of this equation.
- Hence, it cannot be assumed that $x = 4$ will satisfy the equation.
- We will have to simplify the expression on the LHS to see that it reduces to the same form (kx) as that shown on the RHS.

2 **(a)** Solve the equation $\log_3(x-2)=\log_3(12-x)-2$. [4]

(a)

$\log_3(x-2)=\log_3(12-x)-2$

$\log_3(x-2)=\log_3(12-x)-\log_3 3^2$

$\log_3(x-2)=\log_3\left(\frac{12-x}{9}\right)$ (Quotient Law of logarithms)

$x-2=\frac{12-x}{9}$ (By equality of logarithms)

$9(x-2)=12-x$

$10x=30$

$x=3$

Alternative

$\log_3(x-2)=\log_3(12-x)-2$

$\log_3(x-2)-\log_3(12-x)=-2$

$\log_3\left(\frac{x-2}{12-x}\right)=-2$ (Quotient Law of logarithms)

$\frac{x-2}{12-x}=3^{-2}$ (Converting into exponential form)

$9(x-2)=12-x$

$10x=30$

$x=3$

Comments

- A common error is to treat "log" as though it is like a factor:
  $\cancel{\log_3}(x-2)\cancel{=}\cancel{\log_3}(12-x)\cancel{-}\log_3 3^2$
  $x-2=12-x-9.$ ✗
- "log" is a function which gives an output only when written with an input value, e.g. log (x).
- There are two ways which we can *remove* a "log" from an equation:

  1. Equality of log
  $\log_a x=\log_a y \Rightarrow x=y$

  2. Converting into exponential form
  $\log_a x=k \Rightarrow x=a^k$

(b) Given that $(2\log_5 y)(\log_x 5)=8$, express y in terms of x. [3]

(b) $\left(\frac{2\log_x y}{\log_x 5}\right)(\log_x 5)=8$ (Using change of base formula)

$\log_x y=4$

$y=x^4$ (Converting into exponential form)

Alternative

$(2\log_5 y)\left(\frac{\log_5 5}{\log_5 x}\right)=8$ (Using change of base formula)

$2\log_5 y=8\log_5 x$

$\log_5 y=4\log_5 x$

$\log_5 y=\log_5 x^4$ (Power Law of logarithms)

$y=x^4$ (By equality of logarithms)

Comments

- Some students got the result of $y=5^{4\log_5 x}$ or some equivalent form where the power still has a log term. This result can be further simplified.

(c) Express $\mathrm{e}^{4x}+7=3\mathrm{e}^{2x}$ as a quadratic equation in e^{2x} and explain why there are no real solutions. [3]

(c) $\mathrm{e}^{4x}+7=3\mathrm{e}^{2x}$

$(\mathrm{e}^{2x})^2+3\mathrm{e}^{2x}+7=0$ (To express as a quadratic equation in e^{2x})

Let $u=\mathrm{e}^{2x}$.

$u^2+3u+7=0$

Discriminant:

$b^2-4ac=(-3)^2-4(1)(7)$

$=-19<0$

Since **discriminant < 0**,
there are **no real solutions** for the equation.

<u>OR</u>

By using the quadratic formula,

$$u=\frac{-3\pm\sqrt{(-3)^2-4(1)(7)}}{2}$$

$$=\frac{-3\pm\sqrt{-19}}{2}$$

Since $\sqrt{-19}$ is **undefined / does not have any real solution**, the equation has no real solutions.

<u>Comments</u>

- An important law of indices used in this question is $(a^x)^y=a^{xy}$,
  i.e. $\mathrm{e}^{4x}=\left(\mathrm{e}^{2x}\right)^2$.
- Students are expected to provide a clear explanation that links to the conclusion.
- “A negative number cannot be square-rooted” should be phrased as “being undefined” or “having no real solution” to be clearer.

3 The roots of the quadratic equation $2x^2+3x+5=0$ are α and β.

(i) Find the value of $\alpha^2+\beta^2$. [3]

(i) $2x^2+3x+5=0$ has roots α and β

Sum of roots = $\alpha+\beta=-\dfrac{3}{2}$

Product of roots = $\alpha\beta=\dfrac{5}{2}$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

$$=\left(-\frac{3}{2}\right)^2-2\left(\frac{5}{2}\right)$$

$$=-2\frac{3}{4} \quad \text{or} \quad -\frac{11}{4} \quad \text{or} \quad -2.75$$

(ii) Show that the value of $\dfrac{\alpha^3+\beta^3}{\alpha\beta}$ is $\dfrac{63}{20}$. [2]

(ii) $\alpha^3+\beta^3=(\alpha+\beta)[(\alpha^2+\beta^2)-\alpha\beta]$

$$=\left(-\frac{3}{2}\right)\left[\left(-\frac{11}{4}\right)-\frac{5}{2}\right]$$

$$=\frac{63}{8}$$

$$\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{63}{8}\div\frac{5}{2}=\frac{63}{20} \text{ (shown)}$$

(iii) Find a quadratic equation whose roots are $\dfrac{\alpha^2}{\beta}$ and $\dfrac{\beta^2}{\alpha}$. [3]

(iii) Sum of new roots $=\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}=\dfrac{\alpha^3+\beta^3}{\alpha\beta}$

$$=\frac{63}{20}$$

Product of new roots

$$=\frac{\alpha^2}{\beta}\times\frac{\beta^2}{\alpha}$$

$$=\alpha\beta$$

$$=\frac{5}{2}$$

Quadratic equation is $x^2-\dfrac{63}{20}x+\dfrac{5}{2}=0$ or $20x^2-63x+50=0$

Comments

- There were some errors made when giving the quadratic equation such as $x^2-\dfrac{63}{20}x+\dfrac{5}{2}$ without " = 0 " or some slips when multiplying to make all the coefficients integer.

4 **(i)** Show that $\dfrac{\operatorname{cosec}\theta+\sin\theta}{\operatorname{cosec}\theta-\sin\theta}=2\sec^2\theta-1$. [4]

(i)

$$\frac{\operatorname{cosec}\theta+\sin\theta}{\operatorname{cosec}\theta-\sin\theta}=\frac{\dfrac{1}{\sin\theta}+\sin\theta}{\dfrac{1}{\sin\theta}-\sin\theta}$$

$$=\frac{\dfrac{1+\sin^2\theta}{\sin\theta}}{\dfrac{1-\sin^2\theta}{\sin\theta}}$$

$$=\frac{1+\sin^2\theta}{1-\sin^2\theta}$$

$$=\frac{1+\sin^2\theta}{\cos^2\theta}$$

$$=\frac{1+1-\cos^2\theta}{\cos^2\theta} \qquad \text{OR} \qquad =\frac{1}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}$$

$$=\frac{2}{\cos^2\theta}-\frac{\cos^2\theta}{\cos^2\theta} \qquad \text{OR} \qquad =\frac{1}{\cos^2\theta}+\tan^2\theta$$

$$=2\sec^2\theta-1 \qquad\qquad =\sec^2\theta+\sec^2\theta-1$$

$$=2\sec^2\theta-1$$

Comments

- When given reciprocal functions like $\operatorname{cosec}\theta$, $\sec\theta$ or $\cot\theta$, it is advised to express it in terms of $\sin\theta$, $\cos\theta$ or $\tan\theta$.
- Students should then make attempts to simplify the fraction further before determining which identity would be useful to apply towards getting $2\sec^2\theta-1$.
- It is also useful to work with the end in mind, thinking about what function would lead to $\sec^2\theta$.

(ii) Hence, solve the equation $\dfrac{\operatorname{cosec}2x+\sin 2x}{\operatorname{cosec}2x-\sin 2x}=3$ for $0\le x\le\dfrac{\pi}{2}$, leaving your answers in terms of π. [4]

(ii)

$$\frac{\operatorname{cosec}2x+\sin 2x}{\operatorname{cosec}2x-\sin 2x}=3$$

$$2\sec^2 2x-1=3$$

$$\sec^2 2x=2$$

$$\sec 2x=\pm\sqrt{2}$$

$$\cos 2x=\pm\frac{1}{\sqrt{2}}$$

$$\text{Basic angle}=\cos^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{4}$$

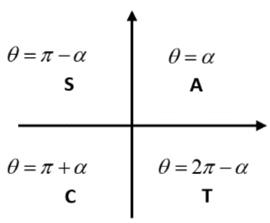
New domain: $0\le x\le\dfrac{\pi}{2}\Rightarrow 0\le 2x\le\pi$

$$2x=\frac{\pi}{4}\text{ or }\frac{3\pi}{4}$$

$$x=\frac{\pi}{8}\text{ or }\frac{3\pi}{8}$$

Comments

- Most students were successful in seeing the relationship between (i) and (ii).
- A common mistake is to only give the positive square root and miss out the **negative square root**. It must be noted that when taking **square roots** to finding an unknown, the square roots could be *positive* or *negative*.
- Students should make it a point to consider the new domain (for 2*x*) so as not to miss out any possible angle that lies in the required quadrant.



5 **(i)** Write down, and simplify, the first 4 terms in the expansion $(1+x)^7$ in ascending powers of x. [2]

(i)

$$(1+x)^7 = 1+\binom{7}{1}x+\binom{7}{2}x^2+\binom{7}{3}x^3...$$
$$= 1+7x+21x^2+35x^3+...$$

(ii) Write down the general term in the binomial expansion of $\left(x^2-\frac{2}{x^3}\right)^9$. [1]

(ii) General Term, $T_{r+1} = \binom{9}{r}x^{2(9-r)}\left(-\frac{2}{x^3}\right)^r$

$$= \binom{9}{r}x^{18-2r}(-2)^r x^{-3r}$$
$$= \binom{9}{r}x^{18-5r}(-2)^r$$

Comments

$$\binom{9}{r}x^{18-2r}(-2)^r x^{-3r} \neq \binom{9}{r}x^{18-5r}(-2^r)$$ ✗

A common error found here is simplifying $(-2)^r$ as (-2^r).

(-2^r) should be understood as -1×2^r which gives a negative value regardless of the value of r. This is not the same as $(-2)^r$.

(iii) Write down the power of x in this general term. [1]

(iii) Power $= 18-5r$

(iv) Hence, or otherwise, determine the coefficient of x^3 in the expansion of $(1+x)^7 + \left(x^2-\frac{2}{x^3}\right)^9$. [2]

(iv) For term in x^3,

$18-5r=3$

$r=3$

Coefficient of $x^3 = 35+\binom{9}{3}(-2)^3 = -637$

6 It is given that $f(x)=3\cos\left(\frac{x}{2}\right)+k$, where k is a constant. The graph of $y = f(x)$ passes through the point $(\pi,-1)$.

(i) State the amplitude and period of f(x). [2]

(i) Amplitude = 3

$$\text{Period} = \frac{2\pi}{\frac{1}{2}}$$

$$= 4\pi$$

(ii) Find the value of k. [2]

(ii) At $(\pi,-1)$

$$-1=3\cos\frac{\pi}{2}+k$$

$$-1=0+k$$

$$k=-1$$

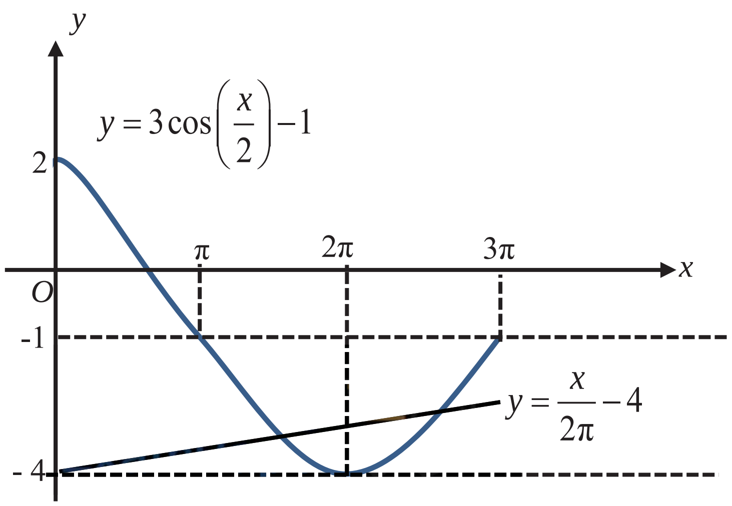
Comments

Students should show clear working when trying to find the value of k.

Evidence of substituting $(\pi,-1)$ into the equation should be seen in an attempt to find k.

(iii) Sketch the graph of $y=f(x)$ for $0\le x\le 3\pi$. [3]

(iii)



Comments

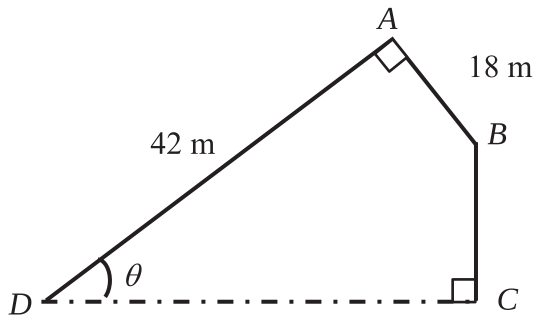
- Note that the graph must be sketched only for $0\le x\le 3\pi$.
- The critical points of the graph (max/min point and points on the axis of the curve) must be indicated.

(iv) By drawing a suitable straight line to the sketch in **(iii)**, find the number of solutions to the equation $f(x)+4=\frac{x}{2\pi}$ for $0\le x\le 3\pi$. [1]

Draw line $y=\frac{x}{2\pi}-4$.

Number of solutions = 2

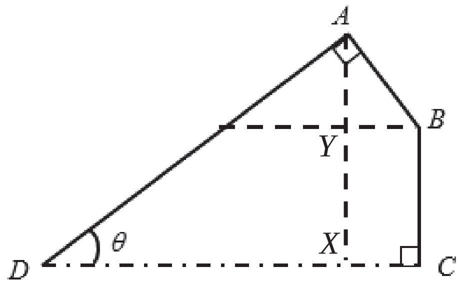
7



The diagram shows a plot of land being fenced up along *AB*, *BC* and *AD*, where *AB* = 18 m, *AD* = 42 m, angle *DAB* = angle *BCD* = 90° and the acute angle *ADC* = θ can vary.

(i) Show that *L* m, the length of the fence, can be expressed as $L = 60 + 42\sin\theta - 18\cos\theta$. [2]

(i)



$\angle DAX = 90° - \theta$

$\angle XAB = \theta$

$AX = 42\sin\theta$

$BC = 42\sin\theta - 18\cos\theta$

$L = AD + AB + BC$

$= 42 + 18 + 42\sin\theta - 18\cos\theta$

$= 60 + 42\sin\theta - 18\cos\theta$

Comments

- Some attempts to "cut the angle θ" were observed. Since the final expression to be proved involves angle θ, it wouldn't be helpful to cut the angle up in any way.
- Instead attempts should be made to cut the diagrams up to get right angle triangles where the sides are related to the length from *A* to *DC*.
- Working must be shown clearly to show the components of the length of fence that result in $L = 60 + 42\sin\theta - 18\cos\theta$.

(ii) Express *L* in the form $p + R\sin(\theta - \alpha)$, where *p* and $R > 0$ are constants and $0° < \alpha < 90°$. [4]

(ii) $42\sin\theta - 18\cos\theta = R\sin(\theta - \alpha)$

$= R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

Comparing coefficients:

$R\sin\alpha = 18$ and $R\cos\alpha = 42$

$\tan\alpha = \dfrac{18}{42}$

$\alpha = 23.1985°$

$R = \sqrt{18^2 + 42^2} = \sqrt{2088} = 6\sqrt{58}$

$L = 60 + 6\sqrt{58}\sin(\theta - 23.2°)$

Comments

- The working on comparing the coefficients is required as evidence of understanding how *R* and α are related to the coefficients.
- Always provide greater accuracy to the intermediate values obtained for *R* and α, where applicable.

  These values are required in the subsequent parts to obtain more accurate results/answers.

(iii) Given that the exact length of the fence used is 92 m, find the value of θ. [3]

(iii)

$$60+6\sqrt{58}\sin(\theta-23.1985°)=92$$
$$6\sqrt{58}\sin(\theta-23.1985°)=32$$
$$\sin(\theta-23.1985°)=\frac{32}{6\sqrt{58}}$$
$$\text{Basic angle} = \sin^{-1}\left(\frac{32}{6\sqrt{58}}\right)=44.4511°$$
$$\theta-23.1985°=44.4511°$$
$$\theta=44.4511°+23.1985°$$
$$\theta=67.6° \quad \text{(to 1 d.p.)}$$

Comments

Check that the angle θ should be acute based on the context.

Hence, students should always make it a point to check that the answer is reasonable / makes sense in relation to the context.

8 Given that $\dfrac{4x^2-x-3}{2x^2-x}=a+\dfrac{bx+c}{2x^2-x}$,

(i) find the value of each of the integers a, b and c. [2]

(i) $\dfrac{4x^2-x-3}{2x^2-x}=\dfrac{2(2x^2-x)+x-3}{2x^2-x}$

$$=2+\frac{x-3}{2x^2-x}$$

$\therefore a=2,\ b=1$ and $c=-3$

Comments

- Students should recognise that the algebraic fraction on the LHS is improper.
- Hence, the expression on the RHS is obtained by either long division or comparing coefficients of an identity.
- It is recommended, as a general approach that, if students are stuck, they should assign random non-zero values to the unknowns to be found. This will allow students to continue with the other parts and get some credit.

(ii) Using the values of b and c obtained in part **(i)**, express $\dfrac{bx+c}{2x^2-x}$ in partial fractions. [4]

(ii) $\dfrac{x-3}{2x^2-x}=\dfrac{x-3}{x(2x-1)}$

$$=\frac{A}{x}+\frac{B}{2x-1}$$

$$=\frac{A(2x-1)+Bx}{x(2x-1)}$$

$x-3=A(2x-1)+Bx$

Let $x=0$,

$0-3=A(-1)$

$A=3$

Let $x=0.5$,

$-2.5=0.5B$

$B=-5$

$\therefore \dfrac{x-3}{2x^2-x}=\dfrac{3}{x}-\dfrac{5}{2x-1}$

Hence, using parts **(i)** and **(ii)**, find

(iii) $\int \frac{8x^2-2x-6}{2x^2-x}\,\mathrm{d}x.$ [4]

(iii)
$$\int \frac{8x^2-2x-6}{2x^2-x}\,\mathrm{d}x = 2\int \frac{4x^2-x-3}{2x^2-x}\,\mathrm{d}x$$
$$= 2\int 2+\frac{3}{x}-\frac{5}{2x-1}\,\mathrm{d}x$$
$$= 2\left(2x+3\ln x-\frac{5}{2}\ln(2x-1)\right)+c$$
$$= 4x+6\ln x-5\ln(2x-1)+c,$$
where c is a constant

<u>Comments</u>

- Some students still have difficulty recognising that integrating functions such as $\frac{3}{x}$ and $-\frac{5}{2x-1}$ gives "ln" functions. To overcome this, students can first recognise that $\frac{3}{x}=3x^{-1}$. Integrating $3x^{-1}$ as a power function yields $\frac{3x^{-1+1}}{-1+1}$ which is undefined.
- Since this involves indefinite integral, there will be an arbitrary constant at the end of the integration. Students should make it a point to check for the constant after obtaining their final answer.

9 A bee leaves its hive and flies towards a flower in a straight path. The velocity, v m/s, that it flies in t s after it leaves the hive is given by

$$v = 4e^{-\frac{t}{5}} - 2.$$

(i) Find the bee's acceleration 3 seconds after it leaves the hive. [3]

(i)

$$a = -\frac{4}{5}e^{-\frac{t}{5}}$$

When $t = 3$,

$$a = -\frac{4}{5}e^{-\frac{3}{5}} = -0.439 \text{ m/s}^2 \text{ (3 s.f.)}$$

Comments

- Some difficulties were observed in differentiating exponential functions which should be based on the rule:
$$\frac{d}{dt}\left(e^{f(t)}\right) = f'(t)e^{f(t)}$$
- Students should consider a two-step process when differentiating $e^{f(t)}$:
Step 1: Differentiate f(t) with respect to t.
Step 2: Multiply this result to the original exponential term to get $f'(t)e^{f(t)}$.

The bee comes to an instantaneous rest when it reaches the flower.

(ii) Show that the time when the bee reaches the flower is $t = \ln 32$. [3]

(ii)

$$v = 4e^{-\frac{t}{5}} - 2$$

$$4e^{-\frac{t}{5}} - 2 = 0$$

$$e^{-\frac{t}{5}} = \frac{1}{2}$$

$$-\frac{t}{5} = \ln\frac{1}{2}$$

$$t = -5\ln\frac{1}{2}$$

$$t = \ln\left(\frac{1}{2}\right)^{-5} = \ln 32$$

Comments

- The key understanding to be shown here is that "instantaneous rest" means "$v = 0$" at a certain time.
- Students must show the use of power law

i.e. $\ln\left(\frac{1}{2}\right)^{-5}$ or $\ln 2^5$ before the full credit can be given.

(iii) Calculate the distance of the flower from the hive. [4]

(iii) $s = \int 4e^{-\frac{t}{5}} - 2 \ \mathrm{d}t$

$= -20e^{-\frac{t}{5}} - 2t + c$

When $t = 0$, $s = 0$.

$0 = -20e^{0} + c$

$c = 20$

$s = -20e^{-\frac{t}{5}} - 2t + 20$

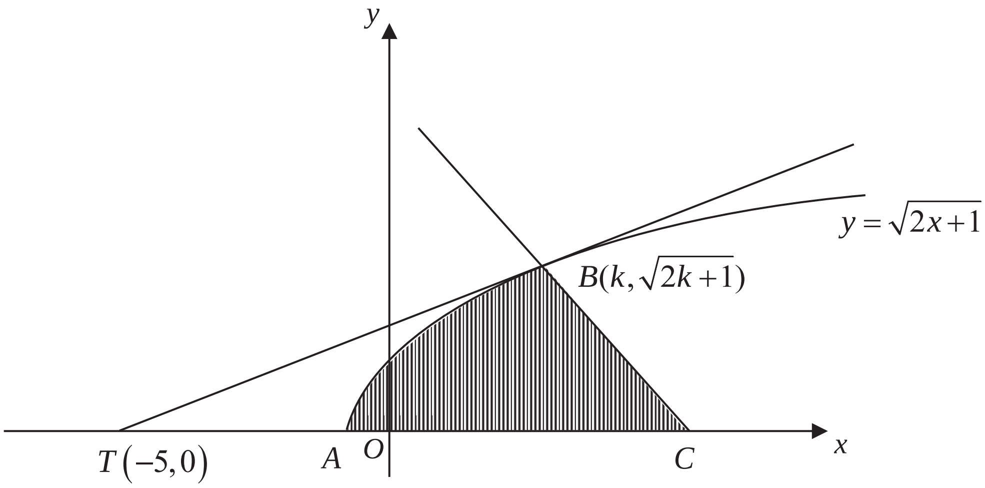
At $t = \ln 32$,

$s = -20e^{-\frac{\ln 32}{5}} - 2(\ln 32) + 20 = 3.07$ m (3 s.f.)

Comments

- To obtain distance, students must recognise the need to integrate the velocity function.
- Some students missed out the arbitrary constant after integration here.
- There were also some slips in the calculation of the constant.
- To check if the answer is reasonable, students can check that at the point when the bee reaches the flower, it took ln 32 seconds (about 3.5 seconds). Some of the answers have distances that go over a few hundred metres! This must be the fastest bee in the world!

10



The diagram shows part of the curve $y=\sqrt{2x+1}$ passing through the point $B(k,\sqrt{2k+1})$, where k is a constant, $k>-\frac{1}{2}$. The curve meets the x-axis at the point A. The tangent to the curve at B meets the x-axis at the point $T(-5,0)$. Another line passing through B meets the $x-$axis at C.

(i) Prove that $k=4$. [5]

(i) $y=\sqrt{2x+1}$

$$\frac{dy}{dx}=\frac{1}{2}\times(2x+1)^{-\frac{1}{2}}\times 2=\frac{1}{\sqrt{2x+1}}$$

At B,

$$\frac{dy}{dx}=\frac{1}{\sqrt{2k+1}}$$

Gradient of $BT=\dfrac{1}{\sqrt{2k+1}}$

$$\frac{\sqrt{2k+1}-0}{k-(-5)}=\frac{1}{\sqrt{2k+1}}$$

$$2k+1=k+5$$

$$k=4 \text{ (proved)}$$

Comments

- Students must learn to consider the significance of B in this question:

  1. It is the point of contact between the curve and the tangent given (solving simultaneous equations)

  2. The gradient at this point on the curve is equals to the gradient of the tangent.

- The gradient should always be considered first since this is the characteristic of a tangent.
- Many students found the working to be tedious when trying to solve simultaneous equations. In some cases, students were working with both x and k in the same equation and somehow, the k became an x in the process. For this approach, it is important to recognise that the solution to the equation is $x=k$ since that is the point of contact.

(ii) Given that the area of triangle *BTC* is $\frac{33}{2}$ units^2, show that the *x*-coordinate of *C* is 6. [2]

(ii) area of triangle $BTC = \frac{33}{2}$ units^2

$$\frac{1}{2}\times\sqrt{9}\times(x+5)=\frac{33}{2}$$

$$x+5=11$$

$$x=6 \text{ (shown)}$$

Comments

- Some students also used the shoelace method to solve this problem.
- When using the shoelace method with unknown coordinates, it is very important that we take extra care to write down the coordinates of the points taken in an **anti-clockwise direction**. Otherwise, we might not be able to obtain the desired result.
- The simpler approach is for students to recognise that the triangle has a side which is parallel to the *x*-axis. In this case, we can simply use the basic formula to find area of a triangle as shown in the solution.

The curve meets the *x*-axis at the point $A\left(-\frac{1}{2},0\right)$.

(iii) Find the area of the shaded region *BCA*. [5]

(iii) Let *X* be a point vertically below *B* on the *x*-axis.

Area of shaded region $BAX = \int_{-\frac{1}{2}}^{4}\sqrt{2x+1}\,dx$

$$=\left[\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}\times 2}\right]_{-\frac{1}{2}}^{4}$$

$$=9 \text{ units}^2$$

Area of triangle $BCX = \frac{1}{2}\times(6-4)\times 3 = 3$ units^2

Area of shaded region $BCA = 9 + 3 = 12$ units^2

Comments

- Students should recognise that the shaded area consists of two regions:
  (1) an area under the curve, and
  (2) an area under a line.

  For (1), we have to integrate the curve with the limits from $-\frac{1}{2}$ to 4.

  For (2), since it is in the shape of a triangle, we need not perform integration to find the area (which some students did).

11 A tangent to a circle at the point $A(-6,12)$ intersects the y-axis at $y=\dfrac{15}{2}$. The normal to the circle at point B has equation $3y+4x=-12$.

(i) Showing all your working, find the equation of the circle. [7]

(i) Tangent passes through $(-6,12)$ and $(0,\frac{15}{2})$

$$\text{Gradient of tangent} = \frac{12-7.5}{-6-0}$$
$$= -\frac{3}{4}$$

$$\text{Gradient of normal} = -1 \div \left(-\frac{3}{4}\right) = \frac{4}{3}$$

Equation of normal at $(-6,12)$

$$y-12=\frac{4}{3}(x-(-6))$$

$$y=\frac{4}{3}x+20 \quad \text{----- (1)}$$

$$3y+4x=-12 \quad \text{----- (2)}$$

Sub (1) in (2):

$$3\left(\frac{4}{3}x+20\right)+4x=-12$$

$$8x=-72$$

$$x=-9$$

$$3y+4(-9)=-12$$

$$y=8$$

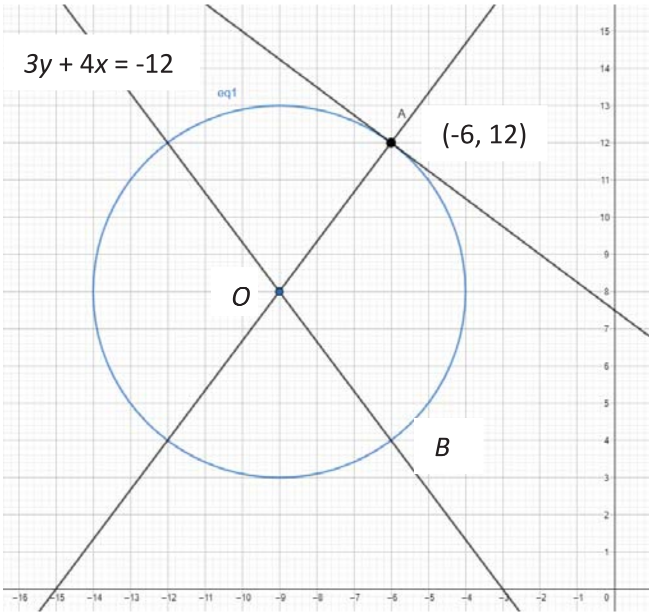
Centre of circle is $(-9,8)$

Radius $=\sqrt{(-6-(-9))^2+(12-8)^2}=\sqrt{25}=5$ units

Equation of circle

$$(x+9)^2+(y-8)^2=5^2$$

$$(x+9)^2+(y-8)^2=25$$



Comments

- Students should recall that to find the equation of a circle, we will require 2 information:
  (1) Centre of the circle,
  (2) Radius of the circle.

- The key understanding is that **any normal to the circle always passes through the centre of the circle**.
- Hence, if we can find two normal, we will be able to solve simultaneous equations to find the centre of the circle.

(ii) The tangent at A meets the x-axis at C. Find the coordinates of C. [2]

(ii) Equation of tangent:

$y=-\frac{3}{4}x+\frac{15}{2}$ since y-intercept is $y=\frac{15}{2}$.

When $y = 0$,

$\frac{3}{4}x=\frac{15}{2}$

$x=10$

C is (10, 0).

(iii) The point B is to the right and below the centre of the circle, and the normal to the circle at B intersects the y-axis at the point D.
Find the distance BD. [3]

(iii) When $x = 0$,

$3y=-12$

$y=-4$

Distance from centre to D

$=\sqrt{(-9-0)^2+(8-(-4))^2}=\sqrt{225}$

=15 units

Since B lies on the circle, distance from B to centre is radius, 5 units

$BD=15-5=10$ units

<u>Comments</u>

- Students should try to provide a sketch for this question in order to see how to best obtain the distance BD.
- In many cases, students try to first find the coordinates of B and then use distance formula to work out BD.
- In this case, it should be recognised that one can taken the difference of the length OD and the radius of the circle to get BD.

BLANK PAGE