BP~600

Name | Reg. No | Class

4EX/5NA

ADDITIONAL MATHEMATICS **4049/01**

Paper 1 [90 marks]

PRELIMINARY EXAMINATION

AUGUST 2023

2 hours 15 minutes

Candidates answer on the Question Paper.

No Additional Materials are required.

INSTRUCTIONS TO CANDIDATES

Do not open this booklet until you are told to do so.

Write your name, register number and class on all the work you hand in.

Write in dark blue or black pen.

You may use a pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **ALL** questions.

Write your answers in the space provided.

If working is needed for any question, it must be shown with the answer.

Omission of essential working will result in loss of marks.

Write the brand and model of your calculator in the space provided below.

INFORMATION FOR CANDIDATES

You are expected to use an electronic calculator to evaluate explicit numerical expressions.

If the degree of accuracy is not specified in the question, and if the answer is not exact, the answer should be given to **three** significant figures. Answers in degrees should be given to **one** decimal place.

For π , use either your calculator value or 3.142, unless the question requires the answer in terms of π .

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is **90**.

Brand / Model of Calculator

This question paper consists of **20** printed pages including **2** blank pages.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\text{Area of } \Delta = \tfrac{1}{2}ab \sin C$$

1 **(i)** Find the range of values of k for which $kx^2 - 4x + 5$ is positive for all real values of x. [2]

(ii) Explain why $\dfrac{x^2 - 2x + 1}{kx^2 - 4x + 5} \geq 0$ for all real values of x for the range of values of k found in part **(i)**. [1]

2 **(a)** Solve the equation $\sqrt{3x-2} = 4 - \sqrt{3x}$ where $x > \dfrac{2}{3}$. [3]

(b) Explain why there is no solution to the equation $\sqrt{3x-2} = 4 + \sqrt{3x}$. [1]

3 **(i)** Prove that $\dfrac{\operatorname{cosec}^2 x - 2}{\operatorname{cosec}^2 x} = \cos 2x$ [4]

3 **(ii)** Hence, find the value of k such that $k\int_{\frac{\pi}{8}}^{\frac{\pi}{2}} \frac{\operatorname{cosec}^2 x - 2}{\operatorname{cosec}^2 x}\,dx = -2\sqrt{2}$. [3]

4 Express $\dfrac{6x^3 - x^2 - 18x - 1}{3x^2 - 5x - 2}$ in the form $Ax + B + \dfrac{C}{3x+1} + \dfrac{D}{x-2}$ where A, B, C and D are constants. [6]

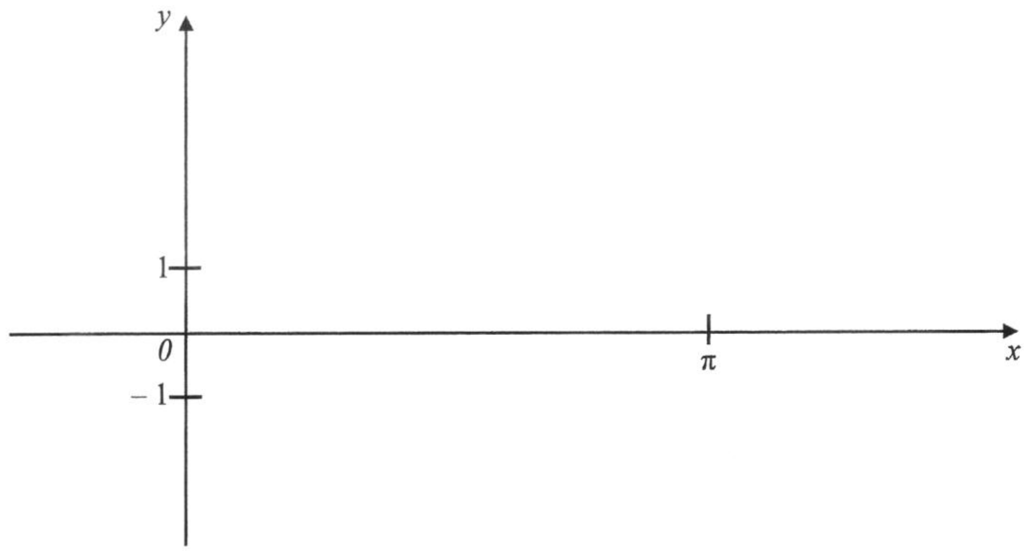
5 The equation of a curve is $y = 2x^3 - 12x^2 + 18x - 7$.

(i) Find the set of values of x for which y is increasing. [3]

(ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y - 18x = 5$. [4]

6 In a wave pool, the height of a wave y meters, can be modelled by $y = 2.3 - \cos 2x$, where x represents the horizontal distance of the wave in metres.

(i) Sketch the graph of $y = 2.3 - \cos 2x$ for $0 \leq x \leq \pi$. [3]



(ii) Solve $2.3 - \cos 2x = 1.5$ for $0 \leq x \leq \pi$. [2]

(iii) A surfer can catch and ride on a wave when it is at least 1.5 m in height.
Find the horizontal distance that the surfer can ride on the wave before it breaks. [2]

7 In an experiment, a liquid is being poured onto a flat surface, forming a circular patch. This circular patch is expanding at a constant rate of 7π cm^2/s.

(i) Find the radius of the circular patch 28 seconds after the liquid is being poured. [2]

(ii) Find the rate of change of the radius at this instant. [3]

(iii) Explain whether the rate of change of the radius will increase or decrease as the experiment continues. [2]

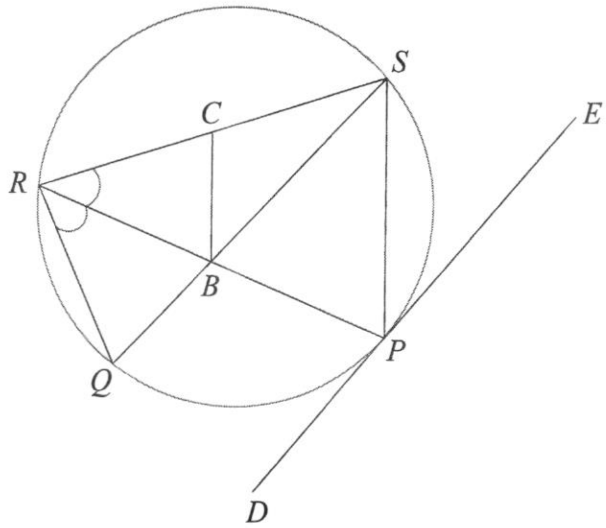
8 **(a)** **(i)** Differentiate $\tan(4x-3)$ with respect to x. [1]

(ii) Explain why the curve $y=\tan(4x-3)$ has no stationary points. [2]

(b) Given that $y=\dfrac{x-2}{\sqrt{x+1}}$, express $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in the form $\dfrac{Ax+B}{2\sqrt{(x+1)^3}}$ where A and B are integers. [4]

8 **(c)** Evaluate $\int_{\frac{\pi}{4}}^{\pi} \left(5\sin 2x + \sec^2 \frac{1}{3}x \right) \mathrm{d}x$. [4]

9



The diagram shows a circle passing through the points P, Q, R and S.

DPE is a tangent to the circle at point P and angle BRC = angle BRQ

B and C are the midpoints of PR and SR respectively.

(i) Prove that angle SPE = angle BRQ. [2]

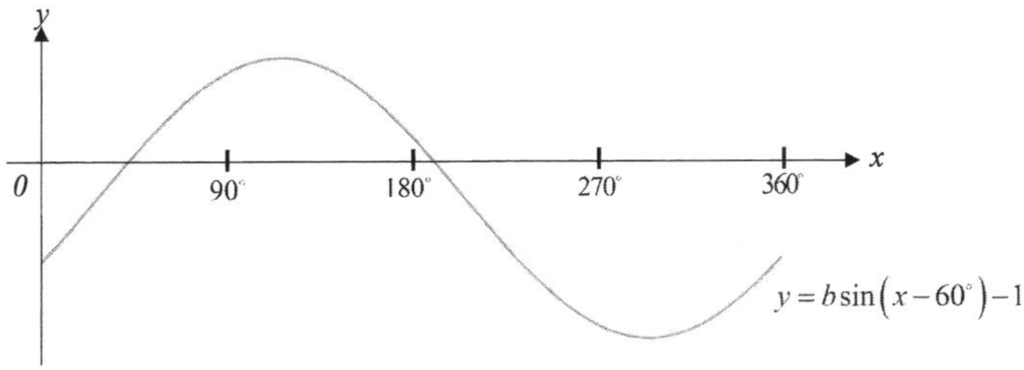
9 **(ii)** Prove that triangle BRC is similar to triangle QRB. [4]

(iii) Show that $BR \times BQ = \frac{1}{2} PS \times QR$. [3]

10 **(a)** Write down the principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ in radians.

Hence, **without using a calculator**, find the exact value of $\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$. [2]

(b)



The diagram above shows the curve with equation $y = b\sin(x - 60^\circ) - 1$ for $0^\circ \le x \le 360^\circ$ and b is a constant. The curve cuts the y-axis at $(0,\ -4)$ and passes through the points $(c,\ 0)$ and $(d,\ 0)$ where c and d are angles measured in degrees.

(i) Show that $b = 2\sqrt{3}$. [3]

10 (b) (ii) Find the value of c and of d. [4]

11 A square has an area $\left(17-4\sqrt{3}\right)$ cm^2. The length of each side of the square can be expressed in the form $\left(a+b\sqrt{3}\right)$ cm, where a and b are integers. Show that $3b^4-17b^2+4=0$. [4]

12 The equation of a curve is $y = x^4 + ax^2 + bx$, where a and b are constants.

(i) Given that $(-1, 5)$ is a stationary point of the curve, find the value of a and of b. [5]

(ii) Determine the nature of the stationary point at $(-1, 5)$. [3]

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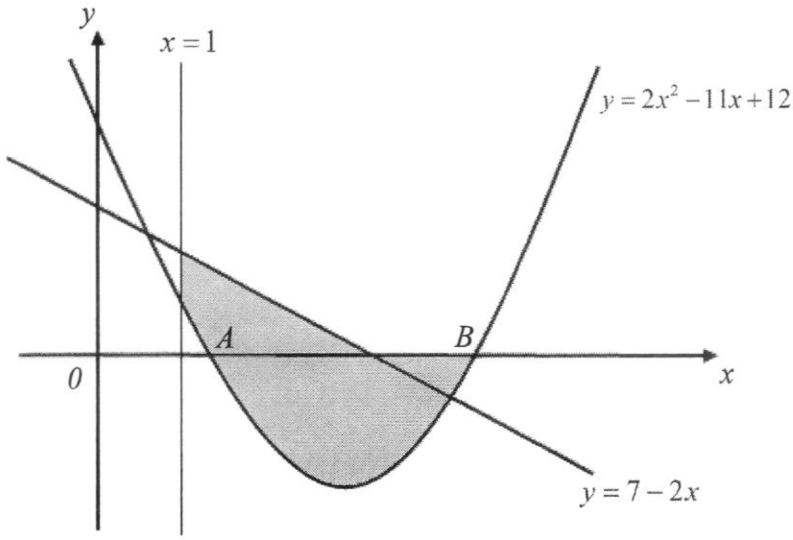

The diagram shows part of a curve $y = 2x^2 - 11x + 12$ meeting the x-axis at points A and B. The diagram also shows the lines $x = 1$ and $y = 7 - 2x$.

(i) Find the x-coordinates of A and B. [2]

13 **(ii)** Hence, find the area of the shaded region. [6]

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Name | Reg. No | Class

MAYFLOWER SECONDARY SCHOOL

4E/5N

ADDITIONAL MATHEMATICS	4049/02
Paper 2 [90 marks]	PRELIMINARY EXAMINATION
	25 August 2023
	2 hours 15 minutes
Candidates answer on the question paper.	

READ THESE INSTRUCTIONS FIRST

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Brand/Model of Calculator

For Examiner's Use	
Total	90

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Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

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$$\sin^2 A + \cos^2 A = 1$$

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$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\text{Area of } \Delta = \frac{1}{2} ab \sin C$$

1 Solve the equation $\log_4 y - \log_{16}(y-2) = \frac{3}{2}\log_4 2$. [5]

2 **(a)** Find the range of values of k for which the line $2x - y = 5$ intersects the curve $xy = kx - 2$ at two distinct points. [4]

(b) Find the smallest integer value of h for which the graph $y = 2x^2 - 4x + h$ lies entirely above the line $y = 3$ for all values of x. [3]

3 **(i)** Show that $\frac{\mathrm{d}}{\mathrm{dx}}(x^2 \ln x) = x + 2x \ln x$. [2]

(ii) Hence, find the value of $\int_1^3 x \ln x \, \mathrm{dx}$, giving your answer in the form $a \ln 3 + b$, where a and b are constants. [5]

4 A particle moves in a straight line such that t seconds after passing a fixed point O, its velocity, v m/s, is given by $v = 2e^{0.1t} - 10e^{-0.3t}$. The particle comes to an instantaneous rest at point A.

(i) Show that the particle reaches A when $t = \frac{5}{2}\ln 5$. [2]

(ii) Find the acceleration of the particle at A. [2]

(iii) Find the distance OA. [4]

(iv) Explain whether the particle is again at O at some instant during the eleventh second after first passing through O. [2]

5 **(i)** In the binomial expansion of $\left(x+\frac{k}{x}\right)^7$, where k is a positive constant, the coefficient of x is 20 times the coefficient of x^5. Show that $k = 2$. [5]

(ii) By considering the general term in the binomial expansion of $\left(x+\frac{k}{x}\right)^7$, explain why there are no even powers of x in this expansion. [1]

(iii) Using the value of k in part **(i)**, find the coefficient of x^7 in the expansion of $\left(1+3x^2\right)\left(x+\frac{k}{x}\right)^7$. [2]

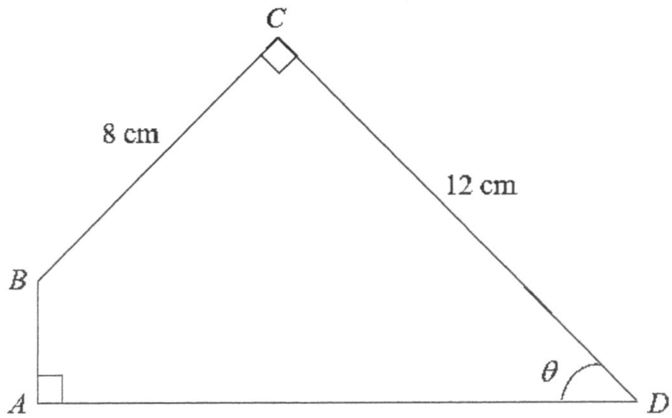
6 The equation of a polynomial is given by $f(x) = ax^3 + x^2 - 13x + b$. f(x) is divisible by $x - 2$ and leaves a remainder of 18 when divided by $x + 1$.

(i) Find the value of a and of b. [4]

(ii) Solve the equation $f(x) = 0$. [4]

(iii) Hence, solve the equation $ax^6 + x^4 - 13x^2 + b = 0$. [2]

7



The diagram shows a quadrilateral *ABCD* where angle *DAB* = angle *BCD* = 90°, angle *CDA* = θ, *BC* = 8 cm and *CD* = 12 cm.

(i) Show that *P* cm, the perimeter of the quadrilateral, is given by

$$P = 4\cos\theta + 20\sin\theta + 20.$$

[3]

(ii) Express P in the form $R\cos(\theta-\alpha)+20$, where $R>0$ and $0°\leq\alpha\leq 90°$. [3]

(iii) State the maximum value of P and the corresponding value of θ. [2]

(iv) Find the value of θ when $P=35$. [2]

8 A curve $f(x)$ is such that $f''(x) = 12x^2 - \frac{2}{(x-1)^2}$ and the point $P(2, 16)$ lies on the curve. The gradient of the curve at P is 34.

(i) Find the equation of the curve. [5]

(ii) Hence explain why the curve is only defined for $x > 1$. [1]

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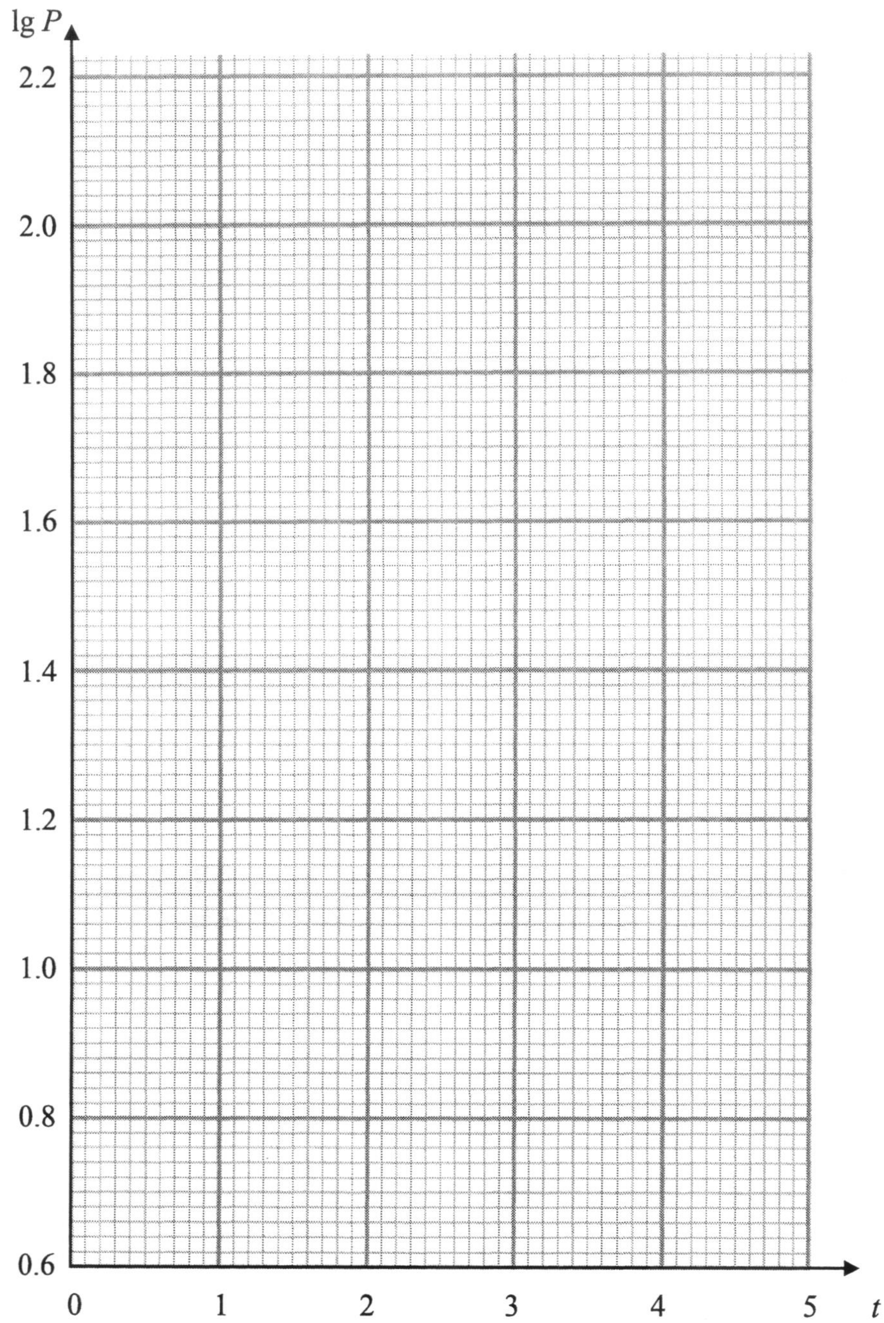
9 The table shows, to 1 decimal place, the population, P, in thousands, of a village t years after 2010.

t years after 2010	1	2	3	4	5
P (in thousands)	10.6	20.0	37.6	70.8	125.9

It is known that P and t are related by the equation $P = ka^t$, where k and a are constants.

(i) On the grid below plot lg P against t and draw a straight line graph. [2]



(ii) Use your graph to estimate the value of k and of a. [4]

(iii) Use your graph to estimate the population of the village two and a half years after 2010. Leave your answer correct to the nearest thousands. [2]

10 **(i)** Find the equation of the line passing through $(3, -2)$ and perpendicular to the line $4y+3x=1$. [2]

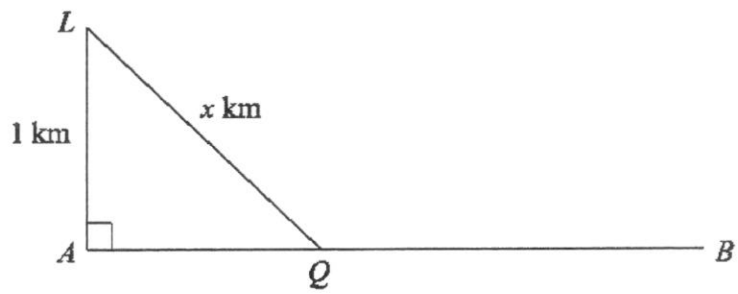
The line $x=11$ is a tangent to the circle C and the equation of the tangent to C at $(3, -2)$ is $4y+3x=1$.

(ii) Given the x-coordinate of the centre of C is k, where k is positive, state the y-coordinate of the centre of C in terms of k.
Hence form an equation in k and show that it reduces to $k^2+3k-54=0$. [4]

(iii) Find the equation of *C*. [3]

(iv) Explain whether the point $(4, -3)$ lies within *C*. [2]

11 The diagram shows a horizontal stretch of a beach AB. Lighthouse L is 1 km away from A and angle $LAQ = 90°$. B is 3 km away from A and $LQ = x$ km. David rows a boat from L to Q at a speed of 3 km/h. He then walks from Q to B at 5 km/h.



(i) Express QB in terms of x. [1]

(ii) Show that the total time, T hours, taken by David to travel from L to B is given by

$$T = \frac{3}{5} + \frac{x}{3} - \frac{\sqrt{x^2 - 1}}{5}.$$ [2]

(iii) Given that x can vary, find the value of x for which the time taken is the least. [5]

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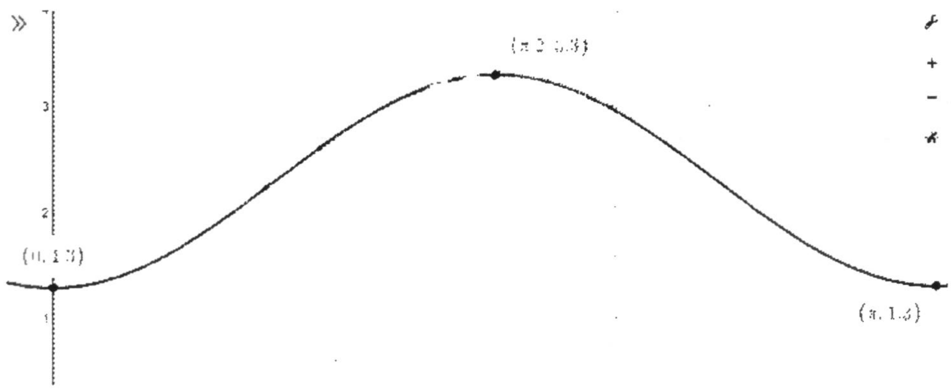
Marking Scheme

Qn No.	Solution
1(i)	$kx^2 - 4x + 5 > 0$ $\Rightarrow \quad b^2 - 4ac < 0$ $\Rightarrow \quad 16 - 4(k)(5) < 0$ $\Rightarrow \quad k > \frac{4}{5}$
1(ii)	$x^2 - 2x + 1 = (x-1)^2 \geq 0 \quad \textit{and} \quad kx^2 - 4x + 5 > 0,$ $\therefore \quad \frac{x^2 - 2x + 1}{kx^2 - 4x + 5} \geq 0 \quad \textit{for all real values of } x.$
2(a)	$3x - 2 = 16 - 8\sqrt{3x} + 3x$ $\frac{-18}{-8} = \sqrt{3x}$ $\sqrt{3x} = \frac{9}{4}$ $3x = \frac{81}{16}$ $x = \frac{27}{16} \quad or \quad 1\frac{11}{16}$
2(b)	$\because \quad \sqrt{3x-2} \quad < \quad \sqrt{3x}$ $\therefore \quad \sqrt{3x-2} \quad < \quad 4 + \sqrt{3x} \quad \textit{for all } x > \frac{2}{3}$ $\Rightarrow \quad$ *There is no solution.*

3(i)	$LHS = \dfrac{1+\cot^2 x - 2}{1+\cot^2 x}$ $= \dfrac{\cot^2 x - 1}{\cot^2 x + 1}$ $= \dfrac{\left(\dfrac{\cos^2 x}{\sin^2 x}\right) - 1}{\left(\dfrac{\cos^2 x}{\sin^2 x}\right) + 1}$ $= \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$ $= \cos^2 x - \sin^2 x$ $= \cos 2x = RHS$
3(ii)	$k \int_{\frac{\pi}{8}}^{\frac{\pi}{2}} \cos 2x \quad dx = -2\sqrt{2}$ $\dfrac{k}{2}\left[\sin 2x\right]_{\frac{\pi}{8}}^{\frac{\pi}{2}} = -2\sqrt{2}$ $\dfrac{k}{2}\left[\sin \pi - \sin \dfrac{\pi}{4}\right] = -2\sqrt{2}$ $-\dfrac{k}{2\sqrt{2}} = -2\sqrt{2}$ $k = 8$

4	*By long division,* $(6x^3 - x^2 - 18x - 1) \div (3x^2 - 5x - 2)$ $= 2x + 3 \quad R(x+5)$ $\dfrac{6x^3 - x^2 - 18x - 1}{3x^2 - 5x - 2} = 2x + 3 + \dfrac{x+5}{(3x+1)(x-2)}$ *Now,* $\dfrac{x+5}{(3x+1)(x-2)} = \dfrac{C}{3x+1} + \dfrac{D}{x-2}$ $\dfrac{x+5}{(3x+1)(x-2)} = \dfrac{C(x-2) + D(3x+1)}{(3x+1)(x-2)}$ *When* $x = -\dfrac{1}{3}$, $C = \dfrac{-\frac{1}{3} + 5}{-\frac{1}{3} - 2} = \dfrac{\left(\frac{14}{3}\right)}{\left(-\frac{7}{3}\right)} = -2$ *When* $x = 2$, $D = \dfrac{2+5}{6+1} = 1$ $\therefore \dfrac{6x^3 - x^2 - 18x - 1}{3x^2 - 5x - 2} = 2x + 3 - \dfrac{2}{3x+1} + \dfrac{1}{x-2}$
5(i)	$\dfrac{dy}{dx} = 6x^2 - 24x + 18$ *For increasing y,* $\dfrac{dy}{dx} > 0$ $\Rightarrow \quad x^2 - 4x + 3 > 0$ $\Rightarrow \quad (x-3)(x-1) > 0$ $\Rightarrow \quad x < 1 \quad or \quad x > 3$

5(ii)	*Gradient of the line* $= 18$ *When* $\frac{dy}{dx} = 18,$ $6x^2 - 24x = 0$ $6x(x-4) = 0$ $x = 0 \quad or \quad x = 4$ $y = -7 \quad or \quad y = 1$ *Coordinates are* $(0,-7)$ *and* $(4,1)$
6(a)	 **G1 for correct shape** **G1 for correct turning point** **G1 for showing one cycle**
6(b)	$\cos 2x = 0.8$ $(basic \angle = 0.6435)$ $2x = 0.6435, \quad 5.6397$ $x = 0.32175, \quad 2.81985$ $\approx 0.322, \quad 2.82$
6(c)	*Hortizontal distance* $= 2.81985 - 0.32175$ $\approx 2.50\ m$

7(i)	$After\ 28\text{sec},\quad area = 7\pi \times 28$ $= 196\pi\ cm^2$ $\therefore\ radius = \sqrt{\frac{196\pi}{\pi}} = 14\ cm$
7(ii)	$A = \pi r^2,\quad \frac{dA}{dr} = 2\pi r$ *By Chain Rule,* $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ $7\pi = 2\pi(14) \times \frac{dr}{dt}$ $\therefore\ \frac{dr}{dt} = \frac{7\pi}{2\pi(14)} = \frac{1}{4}\ cm/s$
7(iii)	$\frac{dr}{dt} = \frac{7\pi}{2\pi r} = \frac{7}{2r}$ $So\ as\ t \to \infty,\quad r\uparrow,\quad and\ \frac{dr}{dt}\downarrow$ *Rate of change of r will decrease.*
8(a)(i)	$\frac{d}{dx}\left[\tan(4x-3)\right] = 4\sec^2(4x-3)$
8(a)(ii)	$\frac{dy}{dx} = \frac{4}{\cos^2(4x-3)},$ $since\ \cos^2(4x-3) > 0,\quad \frac{dy}{dx} > 0\ for\ all\ real\ x.$ $Hence,\ y\ has\ no\ stationary\ points\ as\ \frac{dy}{dx} \neq 0.$

8(b)	$\frac{dy}{dx} = \frac{\sqrt{x+1} - \frac{x-2}{2\sqrt{x+1}}}{x+1}$ $= \frac{2(x+1)-(x-2)}{2\sqrt{(x+1)^3}}$ $= \frac{x+4}{2\sqrt{(x+1)^3}}$
8(c)	$\int_{\frac{\pi}{4}}^{\pi} \left(5\sin 2x \ + \ \sec^2 \frac{1}{3}x\right) dx$ $= \left[-\frac{5}{2}\cos 2x \ + \ 3\tan\frac{x}{3}\right]_{\frac{\pi}{4}}^{\pi}$ $= \left(-\frac{5}{2}\cos 2\pi + 3\tan\frac{\pi}{3}\right) - \left(-\frac{5}{2}\cos\frac{\pi}{2} + 3\tan\frac{\pi}{12}\right)$ $= -\frac{5}{2} \ + \ 3\sqrt{3} \ - \ 0.80385$ $= 1.89230$ ≈ 1.89
9(i)	*Let* $\angle SPE = \angle BRC = a$ (*alternate segment theorem*) *and* $\angle BRC = \angle BRQ = a$ (*given*) $\therefore$ $\angle SPE = \angle BRQ = a$ (*proven*)

9(ii)	$A: \quad \angle BRC = \angle QRB = a \quad (given)$ By Midpoint Theorem, $BC // PS$. Let $\angle RBC = \angle RPS = b \quad (corr. \measuredangle s,\ BC // PS)$ and $\angle RQB = \angle RPS = b \quad (\measuredangle s \text{ in same segment})$ $A: \quad \angle RBC = \angle RQB = b \quad (from\ above)$ $A: \quad \angle RCB = \angle RBQ \quad (\measuredangle \text{ sum of } \Delta)$ Hence, ΔBRC and ΔQRB are similar (AAA).
9(iii)	By Midpoint Theorem, $BC = \frac{1}{2}PS$. Using similar Δs, $\frac{BR}{QR} = \frac{BC}{QB}$ $BR \times QB = BC \times QR$ $BR \times BQ = \frac{1}{2}PS \times QR \quad (shown)$
10(a)	$x = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} \quad (principal\ value)$ $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

10(b)(i)	$At\ (0,-4),\quad -4 = b\sin(0° - 60°) - 1$ $-3 = b\left(-\frac{\sqrt{3}}{2}\right)$ $b = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ $= 2\sqrt{3}\quad (shown)$
10(b)(ii)	*When* $y = 0$, $0 = 2\sqrt{3}\sin(x - 60°) - 1$ $\sin(x - 60°) = \frac{1}{2\sqrt{3}}$ $(basic\ \angle = 16.7787°)$ $x - 60° = 16.7787,\quad 163.2213°$ $x = 76.8°,\quad 223.2°$
11	$(Length)^2 = 17 - 4\sqrt{3}$ $\left(a + b\sqrt{3}\right)^2 = 17 - 4\sqrt{3}$ $a^2 + 2ab\sqrt{3} + 3b^2 = 17 - 4\sqrt{3}$ $\therefore\quad 2ab = -4\quad and\quad a^2 + 3b^2 = 17$ $a = -\frac{2}{b}\qquad \left(-\frac{2}{b}\right)^2 + 3b^2 = 17$ $\frac{4}{b^2} + 3b^2 = 17$ $4 + 3b^4 = 17b^2$ $3b^4 - 17b^2 + 4 = 0\quad (shown)$

12(i)	$y = x^4 + ax^2 + bx$ $\frac{dy}{dx} = 4x^3 + 2ax + b$ $At\left(-1,5\right), \quad 5 = 1 + a - b \Rightarrow a - b = 4 \quad \cdots\cdots(1)$ $and \quad \left.\frac{dy}{dx}\right\rvert_{x=-1} = 0$ $-4 - 2a + b = 0 \Rightarrow b - 2a = 4 \quad \cdots\cdots(2)$ $(1)+(2): \quad -a = 8$ $a = -8$ $b = -12$
12(ii)	$\frac{d^2y}{dx^2} = 12x^2 + 2a$ $= 12x^2 - 16$ $When\ x = -1, \quad \frac{d^2y}{dx^2} = -4 < 0$ $\therefore \quad (-1,5)\ is\ a\ \max.\ point$
13(i)	$2x^2 - 11x + 12 = 0$ $(2x-3)(x-4) = 0$ $x_A = \frac{3}{2} \quad or \quad x_B = 4$

13(ii)	*When* $y=0,\quad 7-2x=0 \;\Rightarrow\; x=\frac{7}{2}$ *When* $x=1,\quad y=7-2=5$ *Shaded region* $=\frac{1}{2}\times\frac{5}{2}\times 5 \;-\; \int_1^{\frac{3}{2}}\left(2x^2-11x+12\right)dx \;+\; \left\lvert\int_{\frac{3}{2}}^{4}\left(2x^2-11x+12\right)dx\right\rvert$ $=\frac{25}{4} \;-\; \left[\frac{2}{3}x^3-\frac{11}{2}x^2+12x\right]_1^{\frac{3}{2}} \;+\; \left\lvert\left[\frac{2}{3}x^3-\frac{11}{2}x^2+12x\right]_{\frac{3}{2}}^{4}\right\rvert$ $=\frac{25}{4} \;-\; \left[\frac{63}{8}-\frac{43}{6}\right] \;+\; \left\lvert\left[\frac{8}{3}-\frac{63}{8}\right]\right\rvert$ $=\frac{25}{4} \;-\; \frac{17}{24} \;+\; \left\lvert-\frac{125}{24}\right\rvert$ $=\frac{43}{4}$ *or* $10.75\ unit^2$

END OF MARKING SCHEME

Sec 4 Express A Math Prelim Paper 2 2023 Marking Scheme

1 Solve the equation $\log_4 y - \log_{16}(y-2) = \frac{3}{2}\log_4 2$.

$$\log_4 y - \log_{16}(y-2) = \frac{3}{2}\log_4 2$$

$$\log_4 y - \frac{\log_4(y-2)}{\log_4 16} = \frac{3}{2}\log_4 2$$

$$\log_4 y - \frac{\log_4(y-2)}{2} = \frac{3}{2}\log_4 2$$

$$\log_4 y^2 - \log_4(y-2) = \log_4 2^3$$

$$\log_4 \frac{y^2}{y-2} = \log_4 8$$

$$y^2 = 8(y-2)$$

$$y^2 - 8y + 16 = 0$$

$$(y-4)^2 = 0$$

$$y = 4$$

2 **(a)** Find the range of values of k for which the line $2x - y = 5$ intersects the curve $xy = kx - 2$ at two distinct points. [4]

(b) Find the smallest integer value of h for which the graph $y = 2x^2 - 4x + h$ lies entirely above the line $y = 3$ for all values of x. [3]

a	$2x - y = 5$ $y = 2x - 5 \quad ---(1)$ $xy = kx - 2 \quad ---(2)$ Sub (1) into (2): $x(2x-5) = kx - 2$ $2x^2 - 5x - kx + 2 = 0$ $2x^2 - (5+k)x + 2 = 0$ $b^2 - 4ac > 0$ $[-(5+k)]^2 - 4(2)(2) > 0$ $25 + 10k + k^2 - 16 > 0$ $k^2 + 10k + 9 > 0$ $(k+1)(k+9) > 0$ $k < -9,\ k > -1$
b	$2x^2 - 4x + h > 3$ $2x^2 - 4x + h - 3 > 0$ $b^2 - 4ac < 0$ $(-4)^2 - 4(2)(h-3) < 0$ $16 - 8h + 24 < 0$ $8h > 40$ $h > 5$ Smallest integer = 6

[Turn over

3 **(i)** Show that $\frac{d}{dx}(x^2 \ln x) = x + 2x \ln x$. [2]

(ii) Hence, find the value of $\int_1^3 x \ln x \, dx$, giving your answer in the form $a \ln 3 + b$, where a and b are constants. [5]

i	$\frac{d}{dx}(x^2 \ln x) = 2x \ln x + x^2(\frac{1}{x})$ $= x + 2x \ln x$
ii	$\int_1^3 x + 2x \ln x \, dx = \left[x^2 \ln x\right]_1^3$ $\int_1^3 x \, dx + 2x \ln x \, dx = 9 \ln 3 - 0$ $\left[\frac{x^2}{2}\right]_1^3 + \int_1^3 2x \ln x \, dx = 9 \ln 3$ $\frac{9}{2} - \frac{1}{2} + \int_1^3 2x \ln x \, dx = 9 \ln 3$ $\int_1^3 2x \ln x \, dx = 9 \ln 3 - 4$ $\int_1^3 x \ln x \, dx = \frac{9}{2} \ln 3 - 2$

4 A particle moves in a straight line such that t seconds after passing a fixed point O, its velocity, v m/s, is given by $v = 2e^{0.1t} - 10e^{-0.3t}$. The particle comes to an instantaneous rest at point A.

(i) Show that the particle reaches A when $t = \frac{5}{2}\ln 5$. [2]

(ii) Find the acceleration of the particle at A. [2]

(iii) Find the distance OA. [4]

(iv) Explain whether the particle is again at O at some instant during the eleventh second after first passing through O. [2]

i	$2e^{0.1t} - 10e^{-0.3t} = 0$ $e^{0.1t} = \frac{5}{e^{0.3t}}$ $e^{0.4t} = 5$ $0.4t = \ln 5$ $t = \frac{5}{2}\ln 5$
ii	$a = 2e^{0.1t}(0.1) - 10e^{-0.3t}(-0.3)$ $a = 0.2e^{0.1t} + 3e^{-0.3t}$ Sub $t = \frac{5}{2}\ln 5$, $a = 1.1962$ $a = 1.20$ m/s^2 (3sf)

iii	$s = \dfrac{2e^{0.1t}}{0.1} - \dfrac{10e^{-0.3t}}{-0.3} + c$ $s = 20e^{0.1t} + \dfrac{100}{3}e^{-0.3t} + c$ Sub $s = 0, t = 0$, $0 = 20 + \dfrac{100}{3} + c$ $c = -\dfrac{160}{3}$ $s = 20e^{0.1t} + \dfrac{100}{3}e^{-0.3t} - \dfrac{160}{3}$ Sub $t = \dfrac{5}{2}\ln 5$, $s = -13.457$ Distance $OA = 13.5$ m (3sf)
iv	Sub $t = 10$, $s = 2.6918$ Sub $t = 11$, $s = 7.9794$ Since the displacement at $t = 10$ s and at $t = 11$ s are both positive, the particle does not pass through O at some instant during the eleventh second.

5 **(i)** In the binomial expansion of $\left(x+\dfrac{k}{x}\right)^7$, where k is a positive constant, the coefficient of x is 20 times the coefficient of x^5. Show that $k=2$. [5]

(ii) By considering the general term in the binomial expansion of $\left(x+\dfrac{k}{x}\right)^7$, explain why there are no even powers of x in this expansion. [1]

(iii) Using the value of k in part **(i)**, find the coefficient of x^7 in the expansion of $\left(1+3x^2\right)\left(x+\dfrac{k}{x}\right)^7$. [2]

---

i

$T_{r+1}=\binom{7}{r}(x)^{7-r}(\frac{k}{x})^r$

For x: $x^{7-r}(x^{-1})^r=x^1$

$r=3$

$T_{3+1}=\binom{7}{3}(x)^{7-3}(\frac{k}{x})^3$

$=35k^3x$

For x^5: $x^{7-r}(x^{-1})^r=x^5$

$r=1$

$T_{1+1}=\binom{7}{1}(x)^{7-1}(\frac{k}{x})^1$

$=7kx^5$

$35k^3=20(7k)$

$k^3-4k=0$

$k(k^2-4)=0$

$k=0$ (rej), $k=-2$ (rej) $k=2$

ii	$T_{r+1} = \binom{7}{r}(x)^{7-r}(\frac{k}{x})^r$ $= \binom{7}{r}(k^r)(x^{7-2r})$ Since 2*r* is an even number and 7 is an odd number, $7-2r$ is an odd number. Hence, there are no even powers of *x* in the expansion.
iii	For $x^7 : x^{7-r}(x^{-1})^r = x^7$ $r = 0$ $T_{0+1} = \binom{7}{0}(x)^{7-0}(\frac{2}{x})^0$ $= x^7$ $(1+3x^2)\left(x+\frac{k}{x}\right)^7 = (1+3x^2)(x^7+14x^5+\ldots)$ $= x^7+42x^7+\ldots$ $= 43x^7+\ldots$ Coefficient of $x^7 = 43$

6 The equation of a polynomial is given by $f(x) = ax^3 + x^2 - 13x + b$. f(x) is divisible by $x - 2$ and leaves a remainder of 18 when divided by $x + 1$.

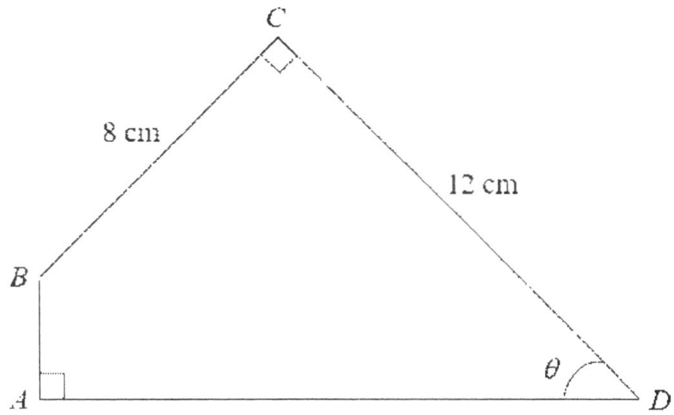
(i) Find the value of a and of b. [4]

(ii) Solve the equation $f(x) = 0$. [4]

(iii) Hence, solve the equation $ax^6 + x^4 - 13x^2 + b = 0$. [2]

i	$8a + 4 - 26 + b = 0$ $b = 22 - 8a \quad ---(1)$ $-a + 1 + 13 + b = 18$ $b = 4 + a \quad ---(2)$ $(1) = (2): \quad 22 - 8a = 4 + a$ $a = 2$ $b = 6$
ii	$2x^2 + 5x - 3$ $x - 2 \overline{)2x^3 + x^2 - 13x + 6}$ $-(2x^3 - 4x^2)$ $5x^2 - 13x + 6$ $-(5x^2 - 10x)$ $-3x + 6$ $-(-3x + 6)$ 0 $(x-2)(2x^2 + 5x - 3) = 0$ $(x-2)(2x-1)(x+3) = 0$ $x = 2, \quad x = \frac{1}{2}, \quad x = -3$
iii	$2x^6 + x^4 - 13x^2 + 6 = 0$ $x^2 = 2, \qquad x^2 = \frac{1}{2}, \qquad x^2 = -3$ (rej) $x = \pm\sqrt{2} \qquad x = \pm\frac{1}{\sqrt{2}}$ $x = \pm 1.41$ (3sf) $x = \pm 0.707$ (3sf)

7



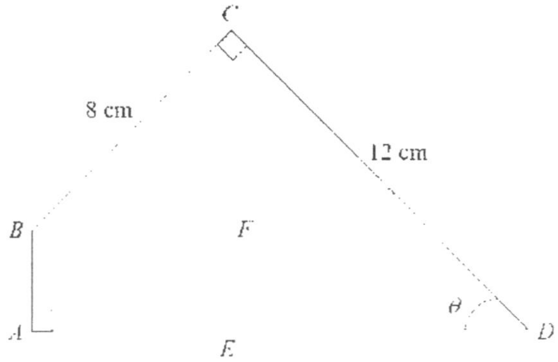
The diagram shows a quadrilateral $ABCD$ where angle DAB = angle BCD = $90°$, angle $CDA = \theta$, BC = 8 cm and CD = 12 cm.

(i) Show that P cm, the perimeter of the quadrilateral, is given by
$$P = 4\cos\theta + 20\sin\theta + 20.$$ [3]

(ii) Express P in the form $R\cos(\theta - \alpha) + 20$, where $R > 0$ and $0° \le \alpha \le 90°$. [3]

(iii) State the maximum value of P and the corresponding value of θ. [2]

(iv) Find the value of θ when $P = 35$. [2]

i	 $\angle BCF = 90-(90-\theta)$ $= \theta$ $\sin\theta = \frac{AE}{8}$ $AE = 8\sin\theta$ $\cos\theta = \frac{ED}{12}$ $ED = 12\cos\theta$ $AB = 12\sin\theta - 8\cos\theta$ $P = 12\sin\theta - 8\cos\theta + 8\sin\theta + 12\cos\theta + 12 + 8$ $P = 4\cos\theta + 20\sin\theta + 20$
ii	$R = \sqrt{4^2 + 20^2}$ $R = \sqrt{416}$ $R = 4\sqrt{26}$ $\alpha = \tan^{-1}\frac{20}{4}$ $\alpha = 78.690$ $P = 4\sqrt{26}\cos(\theta - 78.690) + 20$
iii	Maximum value of $P = 4\sqrt{26} + 20$ $\cos(\theta - 78.690) = 1$ $\theta = 78.690$ $\theta = 78.7°$ (1dp)

iv	$4\sqrt{26}\cos(\theta-78.690)+20=35$ $\cos(\theta-78.690)=0.73543$ $\alpha=42.656$ $\theta-78.690=42.656,\quad 360-42.656-360$ $\theta=121.346$ (rej) $\quad 36.034$ $\theta=36.0°$ (1dp)

8 A curve $f(x)$ is such that $f''(x) = 12x^2 - \dfrac{2}{(x-1)^2}$ and the point $P(2, 16)$ lies on the curve. The gradient of the curve at P is 34.

(i) Find the equation of the curve. [5]

(ii) Hence explain why the curve is only defined for $x > 1$. [1]

i	$f''(x) = 12x^2 - \dfrac{2}{(x-1)^2}$ $= 12x^2 - 2(x-1)^{-2}$ $f'(x) = \dfrac{12x^3}{3} - \dfrac{2(x-1)^{-1}}{-1} + c$ $= 4x^3 + \dfrac{2}{x-1} + c$ Sub $x = 2, f'(x) = 34$, $34 = 4(2^3) + \dfrac{2}{2-1} + c$ $c = 0$ $f'(x) = 4x^3 + \dfrac{2}{x-1}$ $f(x) = \dfrac{4x^4}{4} + 2\ln(x-1) + d$ $= x^4 + 2\ln(x-1) + d$ Sub $x = 2, y = 16$, $16 = 2^4 + 2\ln(2-1) + d$ $d = 0$ $f(x) = x^4 + 2\ln(x-1)$
ii	$\ln(x-1)$ is defined for $x - 1 > 0$. Hence, the curve is only defined for $x > 1$.

9 The table shows, to 1 decimal place, the population, P, in thousands, of a village t years after 2010.

t years after 2010	1	2	3	4	5
P (in thousands)	10.6	20.0	37.6	70.8	125.9

It is known that P and t are related by the equation $P = ka^t$, where k and a are constants.

(i) On the grid below plot lg P against t and draw a straight line graph. [2]

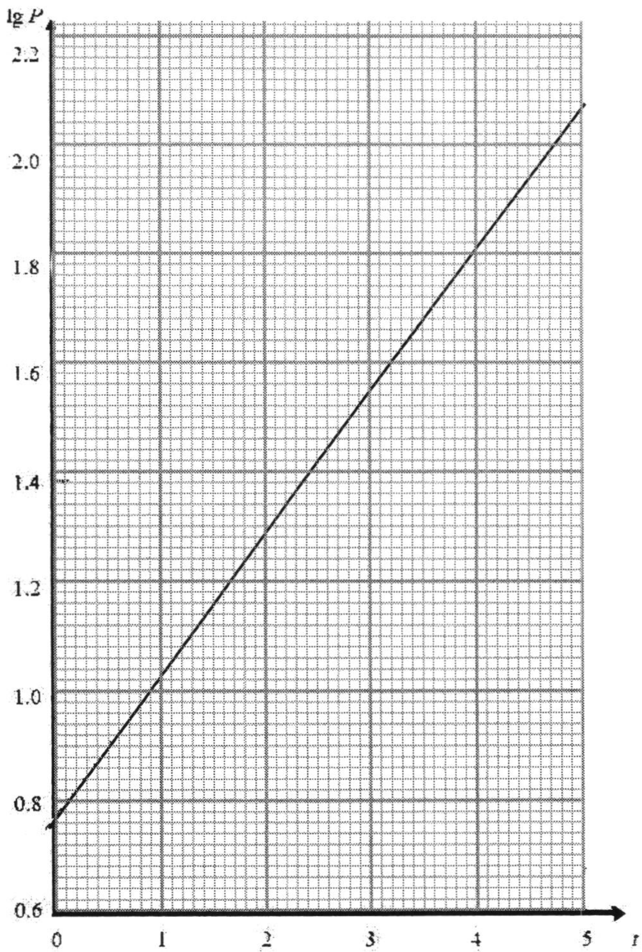
(ii) Use your graph to estimate the value of k and of a. [4]

(iii) Use your graph to estimate the population of the village two and a half years after 2010. Leave your answer correct to the nearest thousands. [2]

lg P

i

t	1	2	3	4	5
lg P	1.03	1.30	1.58	1.85	2.10



ii	$P = ka^t$ $\lg P = \lg(ka^t)$ $\lg P = \lg k + t\lg a$ $\lg P = (\lg a)t + \lg k$ Using $(5, 2.1)$ and $(1.6, 1.2)$ $m = \dfrac{2.1 - 1.2}{5 - 1.6}$ $= 0.26470$ $\lg a = 0.26470$ $a = 1.8395$ $= 1.84$ (3sf) From graph, $c = 0.77$ $\lg k = 0.77$ $k = 5.8884$ $= 5.89$ (3sf)
iii	When $t = 2.5$, $\lg P = 1.44$ $P = 27.5422$ thousand $= 28000$ (nearest thousands)

10 **(i)** Find the equation of the line passing through $(3,\ -2)$ and perpendicular to the line $4y+3x=1$. [2]

The line $x=11$ is a tangent to the circle C and the equation of the tangent to C at $(3,\ -2)$ is $4y+3x=1$.

(ii) Given the x-coordinate of the centre of C is k, where k is positive, state the y-coordinate of the centre of C in terms of k.
Hence form an equation in k and show that it reduces to $k^2+3k-54=0$. [4]

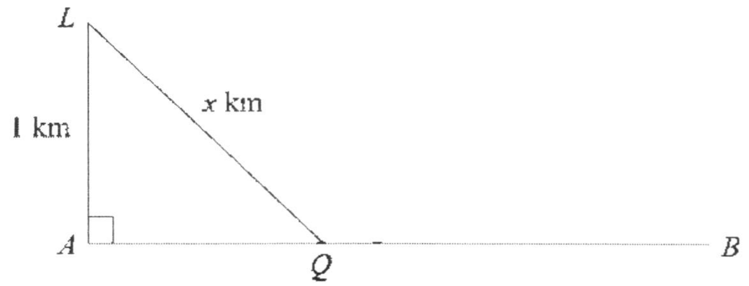
(iii) Find the equation of C. [3]

(iv) Explain whether the point $(4,\ -3)$ lies within C. [2]

i	$4y+3x=1$ $y=-\frac{3}{4}x+\frac{1}{4}$ $m=\frac{4}{3}$ Sub $(3,-2), m=\frac{4}{3}$, $-2=\frac{4}{3}(3)+c$ $c=-6$ $y=\frac{4}{3}x-6$
ii	$y=\frac{4}{3}k-6$ $\sqrt{(k-3)^2+(\frac{4}{3}k-6-(-2))^2}=11-k$ $k^2-6k+9+(\frac{4}{3}k-4)^2=(11-k)^2$ $k^2-6k+9+\frac{16}{9}k^2-\frac{32}{3}k+16=121-22k+k^2$ $\frac{16}{9}k^2+\frac{16}{3}k-96=0$ $16k^2+48k-864=0$ $k^2+3k-54=0$

iii	$k^2+3k-54=0$ $(k-6)(k+9)=0$ $k=6,\quad k=-9$ (rej) Centre $=(k, \frac{4}{3}k-6)$ $=(6,2)$ Radius $=11-k$ $=5$ $C:(x-6)^2+(y-2)^2=25$
iv	Distance of $(4,\ -3)$ from centre $=\sqrt{(6-4)^2+(2-(-3))^2}$ $=\sqrt{29}$ Since the distance of $(4,\ -3)$ from centre > radius of C, the point is outside C.

11 The diagram shows a horizontal stretch of a beach AB. Lighthouse L is 1 km away from A and angle $LAQ = 90°$. B is 3 km away from A and $LQ = x$ km. David rows a boat from L to Q at a speed of 3 km/h. He then walks from Q to B at 5 km/h.



(i) Express QB in terms of x. [1]

(ii) Show that the total time, T hours, taken by David to travel from L to B is given by

$$T = \frac{3}{5} + \frac{x}{3} - \frac{\sqrt{x^2 - 1}}{5}.$$ [2]

(iii) Given that x can vary, find the value of x for which the time taken is the least. [5]

i	$QB = 3 - \sqrt{x^2 - 1}$
ii	$T = \frac{x}{3} + \frac{3 - \sqrt{x^2 - 1}}{5}$ $T = \frac{3}{5} + \frac{x}{3} - \frac{\sqrt{x^2 - 1}}{5}$

iii

$$T = \frac{3}{5} + \frac{x}{3} - \frac{1}{5}(x^2 - 1)^{1/2}$$

$$\frac{dT}{dx} = \frac{1}{3} - \frac{1}{10}(x^2 - 1)^{-1/2}(2x)$$

$$= \frac{1}{3} - \frac{x}{5}\left(\frac{1}{\sqrt{x^2 - 1}}\right)$$

$$\frac{dT}{dx} = 0,$$

$$\frac{1}{3} - \frac{x}{5}\left(\frac{1}{\sqrt{x^2 - 1}}\right) = 0$$

$$\frac{x}{5}\left(\frac{1}{\sqrt{x^2 - 1}}\right) = \frac{1}{3}$$

$$\frac{3x}{5} = \sqrt{x^2 - 1}$$

$$\frac{9}{25}x^2 = x^2 - 1$$

$$\frac{16}{25}x^2 = 1$$

$$x = \frac{5}{4}$$

x	1.15	1.25	1.35
$\frac{dT}{dx}$	< 0	0	> 0

By first derivative test, the time taken is the least when $x = \frac{5}{4}$.

END OF PAPER