Pasir Ris Secondary School

SECONDARY 4 EXPRESS / 5 NORMAL ACADEMIC

PRELIMINARY EXAMINATION 2023

Answer **all** the questions.

1 **(i)** State, in radian, the range of the principal values for $y = \sin^{-1} x$. [1]

(ii) Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. [1]

2 (a) (i) Find the range of values of k for which the graph $y = kx^2 - 4x + k$ lies entirely above the line $y = 3$. [3]

(ii) **Hence**, state the value(s) of k for which the line $y = 3$ is a tangent to the graph $y = kx^2 - 4x + k$. [1]

(b) Find the value of a and of b for which the solution set of $2x^2 < b - ax$ is $-2 < x < 3$. [2]

3 **(a)** Given that $p = 3^a$, express $\frac{27^{a+1}}{9}$ in terms of p.
[3]

(b) By using suitable substitution, solve $e^{2x} - 2e^x - 15 = 0$. [3]

4 A curve has the equation $y = 3e^{\frac{1}{2}x^2 - 2x}$.

(a) Find an expression for $\frac{dy}{dx}$. [1]

(b) Find an expression for $\frac{d^2y}{dx^2}$.
[2]

(c) **Hence,** explain whether the gradient function is an increasing or decreasing function for all values of x. [2]

5 **(a)** Write down and simplify the first 3 terms in the expansion, in ascending powers of x, of $\left(\frac{1}{2x^2}+3x\right)^8$.

[2]

(b) Explain why there is no independent term for $\left(\frac{1}{2x^2}+3x\right)^8$. [3]

(c) **Hence**, find the term independent of x in the expansion of $\left(\frac{1}{2x^2}+3x\right)^8\left(5x^3+\frac{1}{x^2}\right)$.
[4]

6 **(a)** Differentiate $y = \ln\sqrt{\dfrac{3-2x}{5x+3}}$, with respect to x. [3]

(b) Hence, show that $y = \ln\sqrt{\frac{3-2x}{5x+3}}$ has no stationary points for $-\frac{3}{5} < x < \frac{3}{2}$. [3]

7 **(a)** Express $\dfrac{5x+4}{(1+2x)(x+2)^2}$ as partial fractions. [4]

(b) **Hence**, find $\int \frac{5x+4}{(1+2x)(x+2)^2}\,dx$.

[3]

8 Elvin bought a new car and the value, $\$V$, of the car is given by the formula, $V = 130000e^{-mt}$ where m is a constant and t is the age of the car in **months**.
The value of the car after 3 years is expected to be $100 000.

(i) Find the amount that Elvin paid for the car? [1]

(ii) Calculate the value of the car after 65 months. Giving your answer to the nearest dollars. [3]

(iii) In which month did the value of the car first depreciate to $30,000.
Giving your answer to the nearest month. [3]

9 The equation of a circle C is $x^2 + y^2 - 4x - 6y + 4 = 0$.

(i) Find the coordinates of its centre and the radius of the circle. [3]

(ii) Explain whether the point $A(4,5)$ lies outside the circle, inside the circle or on the circle. [2]

(iii) Find the equation of circle D, which is a reflection of circle C along the y-axis. [2]

10

(a) Without using calculator, simplify $\frac{\sqrt{162}+\sqrt{72}}{\sqrt{54}}$. [2]

(b) The area of a rectangular plot of land is $(31+8\sqrt{2})$ cm^2.

The length of the rectangular land is $(3+\sqrt{2})^2$ cm.

Find the breadth of the rectangular land in the form $a+\sqrt{b}$, where a and b are integers. [4]

11 In a particular probiotic drink, it contains live and active "good bacteria".
The population y, in millions, is recorded in a laboratory experiment for 20 minutes.

It is given by $y = Ax^n + 8$ where x is the time measured in minutes, A and n are constants.
The results are shown in the table below.

x	4	8	12	16	20
y	292	1138	8118	17914	35735

(a) Using the graph grid on the following page,

Plot $\lg(y-8)$ against $\lg x$, draw a straight line graph. [3]

(b) Use your graph to estimate the

(i) correct value of y for which an error has been made. [2]

(ii) value of A and of n, [3]

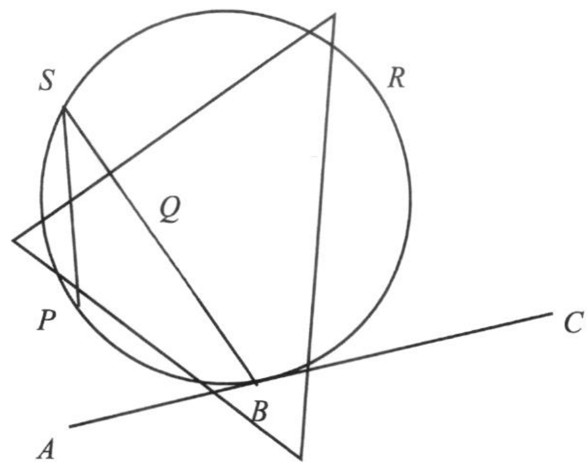
12 **(a)** Show that $\cos^2 52.5° - \sin^2 52.5° = \cos 105°$. [1]

(b) **Without using calculator**, find the value of $\cos^2 52.5° - \sin^2 52.5°$. [5]

13 **(a)** Prove the identity $\sin^2 2x\left(\cot^2 x - \tan^2 x\right) = 4\cos 2x$. [3]

(b) **Hence**, evaluate $\int_0^{\frac{\pi}{2}} \left[\sin^2 2x(\cot^2 x - \tan^2 x) + \tan^2 2x\right] \mathrm{d}x$. [4]

14 In the diagram below, AC is a tangent to the circle at B. SB is a chord of the circle. Given that BP and BR bisects angle SBA and SBC respectively.



(i) Prove that $\angle PBR = 90°$. [3]

(ii) Explain why PR is the diameter of the circle. [1]

(iii) Show that ΔSPQ is similar to ΔRBQ.
[2]

It is given that $4PQ = QR$ and $SQ = QB$.

(iv) Prove that $\left(\frac{QB}{PQ}\right)^2 = 4$. [2]

- End of Paper -

BP~437

Pasir Ris Secondary School

SECONDARY 4 EXPRESS / 5 NORMAL ACADEMIC

PRELIMINARY EXAMINATION 2023

ADDITIONAL MATHEMATICS **4049/02**

25 August 2023

Paper 2

2 hours 15 minutes

Friday 0800-1015

Additional Materials: Electronic calculator

READ THESE INSTRUCTIONS FIRST

Write your name, class and register number on all the work you hand in.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

If working is needed for any question, it must be shown with the answer.
Omission of essential working will result in loss of marks.

You are expected to use a scientific calculator to evaluate explicit numerical expressions.
If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures. Give answers in degrees to one decimal place.

For π, use either your calculator value or 3.142, unless the question requires the answer in terms of π.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is **90**.

This document consists of **24** printed pages.

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx = c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

Answer **all** the questions.

1 **(a)** Sketch the graph of $y = \log_5 x$, indicating the coordinates of the point where the curve $y = \log_5 x$ cuts the x-axis. [2]

(b) Given that $\lg(7y+2) - x^2 = 2$, express y in terms of x. [3]

(c) Solve the equation $\log_3(2x^2+27)-2\log_9(x+12)=1$. [4]

2 The expression $3x^3 + ax^2 + bx + 2$, where a and b are constants, has a factor of $x - 1$. When the expression is divided by $x - 2$, it leaves a remainder that is 2.5 times the remainder when divided by $x + 1$.

(a) Show that $a = 2$ and $b = -7$. [5]

(b) Hence, **by using long division**, solve the equation $3x^3 + ax^2 + bx + 2 = 0$. [3]

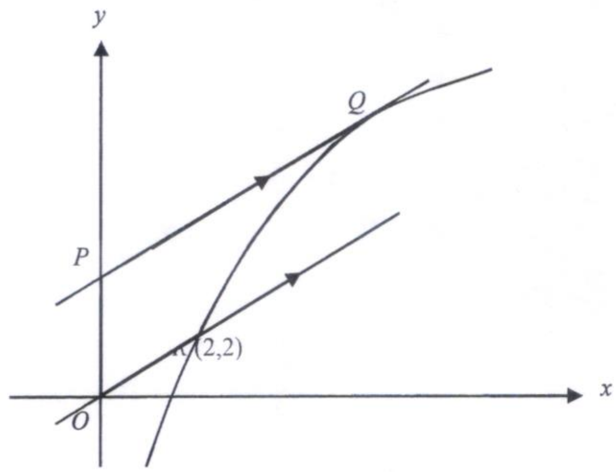
3 The equation of a curve is $y = 1 - 3\cos 2x$ for $0 \le x \le 2\pi$.

(a) State the period of f. [1]

(b) State the maximum and minimum values of y. [2]

(c) Sketch the graph of $y = 1 - 3\cos 2x$ for $0 \le x \le 2\pi$, indicating the number of points of intersections with the axes clearly. [2]

4



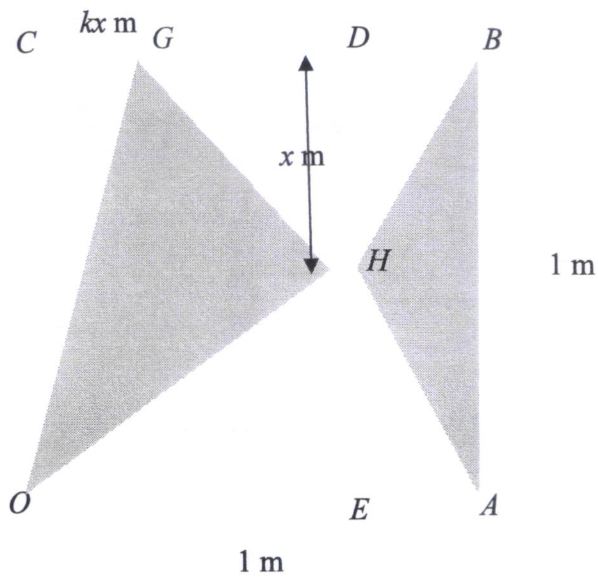
The diagram shows part of the curve and two parallel lines *OR* and *PQ*.

The line *OR* intersects the curve at the point $R(2,2)$ and the line *PQ* is a tangent to the curve at the point *Q*.

(a) Show that the coordinates of *P* and *Q* are $(0,4)$ and $(4,8)$ respectively. [4]

(b) Find the area of the shaded region *OPQR*. [6]

5



In the diagram, *OABC* is a square of side 1m. *D* and *E* are points on *BC* and *AO* respectively. *DE* is parallel to the sides of *CO* and *BA*. *H* is a point on *DE* and *G* is a point on *CB* such that *DH* is x m and *CG* is kx m, where k is a constant and $0 < k < 1$.

(a) Show that the sum, S m^2, of areas of triangles *OHG* and *ABH*, is given by

$$S = \frac{1}{2}\left(1 - kx + kx^2\right).$$

[3]

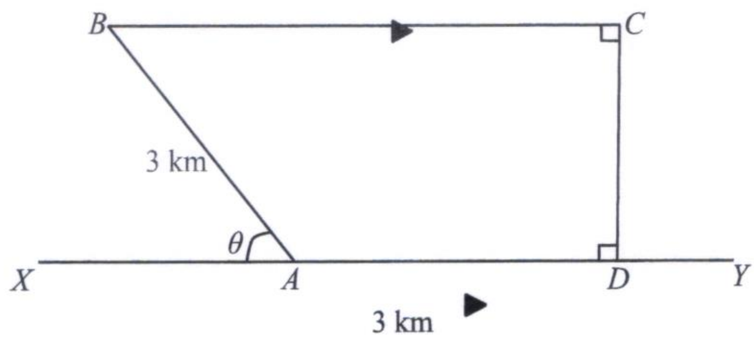
(b) Find the value of x such that S a minimum. [3]

(c) Given that x can vary, find in terms of k, the minimum value of S. [1]

6 **(a)** It is given that $\sin\theta = \frac{1}{\sqrt{1+a^2}}$ and θ is acute. Obtain an expression, in terms of a, for $2\tan\theta$. [2]

(b) Solve the equation $2\sec A = \tan^2 A - 2$ for $0° < A < 360°$ [4]

7 Charles signed up for a race and was given a brochure showing the race route.



XY is a straight road. Participants would start running from point A to D, then from D to C, followed by C to B and finally from B back to A. BC is parallel to XY. CD is perpendicular to both BC and XY. $AB = AD = 3$ km and angle XAB is θ °. The total distance of the route is L km.

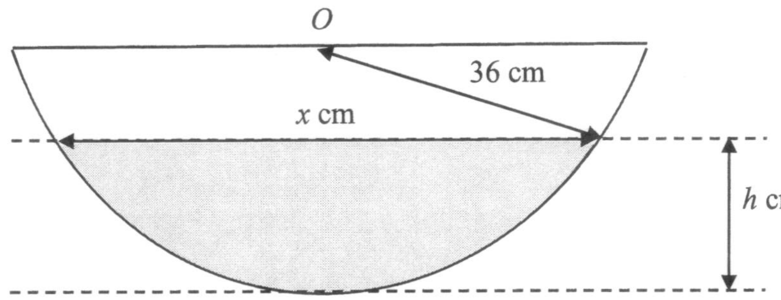
(a) Show that L can be expressed as $p\cos\theta + q\sin\theta + r$, where p, q and r are constants. [2]

(b) Express L in the form $R\cos(\theta-\alpha)+r$, where $R>0$ and α is an acute angle. [3]

(c) The total length of the route is found to be 13 km. Find the possible values of θ. [4]

(d) Charles claims that he can finish the race in under 49 minutes if he maintains his speed of 16 km/h throughout the race regardless of the value of θ. Is Charles's claim true? Explain your answer. [2]

8



The diagram shows the cross section of a hemispherical container with centre O. The container is filled with water and has a radius of 36 cm. Water is leaked through a hole at the bottom of the container such that when the water in the container is at a depth of h cm, the horizontal water surface has a length of x cm.

(a) Show that $x = \sqrt{288h - 4h^2}$. [2]

(b) Given that the length of the horizontal water surface is decreasing at a rate of 4 cm/min, find the rate at which the depth of the water is decreasing when $h = 10$ cm. [5]

9 A particle P moves along a horizontal straight line so that its displacement, s m, from a fixed point O, t seconds after motion has begun is given by $s = 14 + 5t - t^2 - t^3$.

(a) Find an expression, in terms of t, the velocity of P. [1]

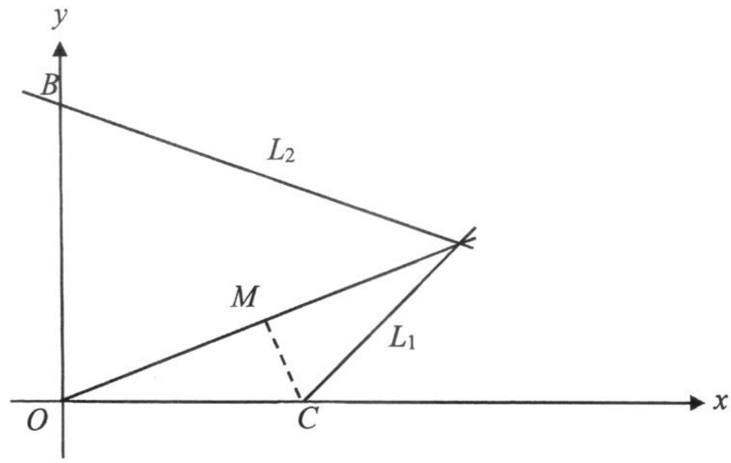
(b) Find the acceleration of P when it is instantaneously at rest. [4]

A second particle Q moves along the same horizontal straight line as P and starts from O at the same instant when P begins to move. The initial velocity of Q is 6 m/s and its acceleration, a m/s^2, t seconds after motion has begun, is given by $a = 4 - 6t$.

(c) Find the value of t at the instant when P and Q collide. [5]

(d) Determine whether P and Q are travelling in the same direction at this instant. [1]

10

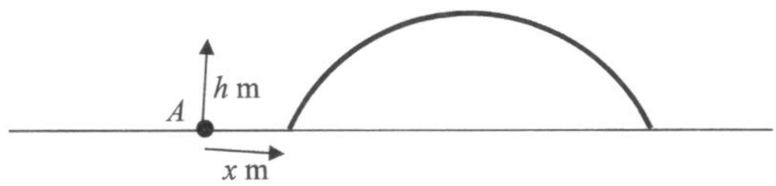


The diagram shows lines L_1 and L_2 intersecting at $A(16, k)$. Line L_1 has equation $3y = 4x - 40$. The point M is the midpoint of OA. The line L_2 has a y-intercept of 48 and the line L_1 intersects the x-axis at C.

(a) Show that angle OMC is $90°$. [5]

(b) Find the area of $OBAC$. [3]

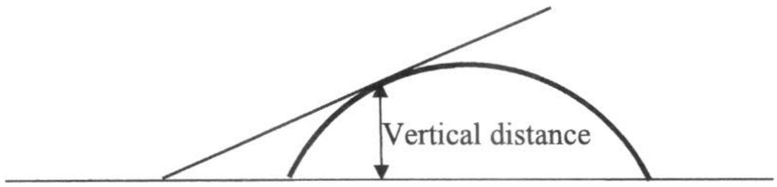
11 A skateboard park has a mould that is h metres high at the point where the horizontal distance, from a fixed point A, is x metres.



The height of the mould can be modelled by $h = -x^2 + 6x - 3$ for $1 < x < 5$.

(a) By **only completing the square**, find the height of the mould at its **highest** point. [2]

(b) A ramp built up the side of the mould is a tangent to the mould.



The ramp can be modelled by the function $h = 2.5x + c$.

Find the vertical distance where the ramp will meet the mould. (*Ignore the thickness of the ramp.*) [3]

(c) The height h metres of a skateboard path at a horizontal distance r metres from another point B, can be modelled by the function $h = \frac{2}{3}r^3 - 4r^2 + 6r$, $0 < r < 4$ For the safety of the skaters, there is a height regulation that requires no part of the skateboard path to be 3 metres above the ground.

Fully describe this curve, including its **turning points**, the **horizontal distance *r* metres from point *B***, and state whether the skateboard path complies with the height regulations. [3]

END OF PAPER

Answer **all** the questions.

1 **(i)** State, in radian, the range of the principal values for $y = \sin^{-1} x$.
[1]

Principal value for $y = \sin^{-1} x$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. [B1]

(ii) Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.
[1]

Let $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\cos y = \frac{\sqrt{3}}{2}$

$\cos y = \cos\left(\frac{\pi}{6}\right)$

Then principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$. [B1] Radian or degree

2 **(a) (i)** Find the range of values of k for which the graph $y = kx^2 - 4x + k$ lies entirely above the line $y = 3$.

[3]

$$kx^2 - 4x + k = 3$$
$$kx^2 - 4x + k - 3 = 0$$

Since the graph lies entirely above the line, $D < 0$ [M1] State $D < 0$

$$(-4)^2 - 4(k)(k-3) < 0$$

$$16 - 4k^2 + 12k < 0$$
$$4k^2 - 12k - 16 > 0$$ [M1 cao] Correct quad inequality

$$(k-4)(k+1) > 0$$

$k < -1$ or $k > 4$ [A1]

(Reject $k < -1$ since $k > 0$)

(ii) Hence, state the value(s) of k for which the line $y = 3$ is a tangent to the graph $y = kx^2 - 4x + k$.

[1]

$k = -1$ or $k = 4$

(b) Find the value of a and of b for which the solution set of $2x^2 < b - ax$ is $-2 < x < 3$. [2]

Since $(x+2)(x-3) < 0$ [M1 cao]

$x^2 - 3x + 2x - 6 < 0$

$x^2 - x - 6 < 0$

$2x^2 < 12 + 2x$

Therefore, $a = -2, b = 12$ [A1]

3 **(a)** Given that $p = 3^a$, express $\frac{27^{a+1}}{9}$ in terms of p. [3]

$\frac{27^{a+1}}{9} = \frac{3^{3(a+1)}}{3^2}$ [M1] Change of base

$= 3^{3a+3-2}$ [M1] Apply any indices law

$= 3^{3a+1}$

$= (3^a)^3(3)$

$= 3p^3$ [A1]

(b) By using suitable substitution, solve $e^{2x} - 2e^x - 15 = 0$. [3]

$e^{2x} - 15 = 2e^x$

$e^{2x} - 2e^x - 15 = 0$

Let $y = e^x$

$y^2 - 2y - 15 = 0$ [M1] Substitution method

$(y-5)(y+3) = 0$

$y = 5$ or $y = -3$

$e^x = 5$ $e^x = -3$ (no solution) [A1]

$x = \ln 5$

$x = 1.61$ (3 s. f.) [A1]

4 A curve has the equation $y = 3e^{\frac{1}{2}x^2 - 2x}$.

(a) Find an expression for $\frac{dy}{dx}$. [1]

$\frac{dy}{dx} = 3e^{\frac{1}{2}x^2 - 2x}(x-2)$ [B1]

(b) Find an expression for $\frac{d^2y}{dx^2}$.
[2]

$$\frac{dy}{dx} = 3e^{\frac{1}{2}x^2-2x}(x-2)$$

$$\frac{d^2y}{dx^2} = (x-2)3e^{\frac{1}{2}x^2-2x}(x-2) + (1)(3e^{\frac{1}{2}x^2-2x})$$

[M1] Apply Product Rule correctly based on part (a)

$$\frac{d^2y}{dx^2} = 3e^{\frac{1}{2}x^2-2x}\left[(x-2)^2+1\right]$$

[A1]

(c) **Hence,** explain whether the gradient function is an increasing or decreasing function for all values of x. [2]

$$\frac{d^2y}{dx^2} = 3e^{\frac{1}{2}x^2-2x}\left[(x-2)^2+1\right]$$

Since $3e^{\frac{1}{2}x^2-2x} > 0$, $(x-2)^2 > 0$ and thus $\left[(x-2)^2+1\right] > 0$, therefore $\frac{d^2y}{dx^2} > 0$ [M1]

Therefore, gradient function of y is an increasing function. [A1]

5 **(a)** Write down and simplify the first 3 terms in the expansion, in ascending powers of x, of $\left(\frac{1}{2x^2}+3x\right)^8$.

[2]

$$\left(\frac{1}{2x^2}+3x\right)^8=\binom{8}{0}\left(\frac{1}{2x^2}\right)^8(3x)^0+\binom{8}{1}\left(\frac{1}{2x^2}\right)^7(3x)+\binom{8}{2}\left(\frac{1}{2x^2}\right)^6(3x)^2+\ldots$$

[M1] Expand all terms correctly

$$=\frac{1}{256x^{16}}+(8)\left(\frac{1}{128x^{14}}\right)(3x)+(28)\left(\frac{1}{64x^{12}}\right)(9x^2)+\ldots$$

$$=\frac{1}{256x^{16}}+\frac{3}{16x^{13}}+\frac{63}{16x^{10}}+\ldots\ldots$$ [A1]

(b) Explain why there is no independent term for $\left(\frac{1}{2x^2}+3x\right)^8$. [3]

The general term is $T_{r+1}=\binom{8}{r}\left(\frac{1}{2x^2}\right)^{8-r}(3x)^r=\binom{8}{r}\left(\frac{1}{2}\right)^{8-r}\left(x^{-2}\right)^{8-r}\left(3^r\right)\left(x^r\right)$

[M1 cao] General Term

$$\left(x^{-2}\right)^{8-r}\left(x^r\right)=x^0$$ [M1] Equate to x^0

$$x^{-16+2r}\left(x^r\right)=x^0$$

$$x^{3r-16}=x^0$$

$$r=5\frac{1}{3}$$

Since the value of r is not an integer, therefore the term x^0 does not exist. Therefore there is no independent term for $\left(\frac{1}{2x^2}+3x\right)^8$. [A1] With reason

(c) **Hence**, find the term independent of x in the expansion of $\left(\frac{1}{2x^2}+3x\right)^8\left(5x^3+\frac{1}{x^2}\right)$. [4]

From **part (b)**,

the general term is $T_{r+1}=\binom{8}{r}\left(\frac{1}{2x^2}\right)^{8-r}(3x)^r=\binom{8}{r}\left(\frac{1}{2}\right)^{8-r}\left(x^{-2}\right)^{8-r}\left(3^r\right)\left(x^r\right)$

For the term x^2, $\left(x^{-2}\right)^{8-r}\left(x^r\right)=x^2$ [M1] Use General Term equate to x^2

$x^{-16+2r}\left(x^{r}\right)=x^{2}$

$x^{3r}=x^{18}$

$r=6$ [M1] Value of $r=6$

The term x^{2} in $\left(\frac{1}{2x^{2}}+3x\right)^{8}$ is $T_{6+1}=\binom{8}{6}\left(\frac{1}{2x^{2}}\right)^{8-6}(3x)^{6}=5103x^{2}$ [M1] Finding x^{2} term or prove no x^{-3} term

$\left(x^{-2}\right)^{8-r}\left(x^{r}\right)=x^{-3}$

$x^{-16+2r+r}=x^{-3}$

$x^{3r-16}=x^{-3}$

$r=\frac{13}{3}$

Since r is not a positive integer value, there is no x^{-3} term.

Hence, the term independent of x is

$\left(\ldots 5103x^{2}+\ldots\right)\left(5x^{3}+\frac{1}{x^{2}}\right)$

$=\ldots 5103\ldots$ [A1]

6 **(a)** Differentiate $y=\ln\sqrt{\frac{3-2x}{5x+3}}$, with respect to x. [3]

$y=\ln\sqrt{\frac{3-2x}{5x+3}}$

$y=\frac{1}{2}[\ln(3-2x)-\ln(5x+3)]$

[M1 cao] Apply logarithm Law

$\frac{dy}{dx}=\frac{1}{2}\left[\frac{-2}{3-2x}-\frac{5}{5x+3}\right]$

[M1 Differentiate either $\ln(3-2x)$ or $\ln(5x+3)$ correctly

$\frac{dy}{dx}=\frac{1}{2}\left[\frac{-2}{3-2x}-\frac{5}{5x+3}\right]$

$\frac{dy}{dx}=\frac{-1}{3-2x}-\frac{5}{2(5x+3)}$ [A1]

(b) Hence, show that $y=\ln\sqrt{\frac{3-2x}{5x+3}}$ has no stationary points for $-\frac{3}{5}<x<\frac{3}{2}$. [3]

Since $\frac{dy}{dx}=\frac{-1}{3-2x}-\frac{5}{2(5x+3)}$

$$= \frac{1}{2x-3} - \frac{5}{2(5x+3)}$$

$$= \frac{2(5x+3)-5(2x-3)}{2x-3}$$

$$= \frac{21}{2(5x+3)(2x-3)}$$ [M1]

$-\frac{3}{5} < x < \frac{3}{2}$, $(5x+3) > 0,\ (2x-3) < 0$, $\frac{21}{2(5x+3)(2x-3)} < 0$ [M1]

Since $\frac{dy}{dx} \neq 0$, therefore there is no stationary points for y. [A1] With Reason

7 **(a)** Express $\frac{5x+4}{(1+2x)(x+2)^2}$ as partial fractions. **[4]**

Let $\frac{5x+4}{(1+2x)(x+2)^2} = \frac{A}{1+2x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ [M1 cao]

$5x+4=A(x+2)^2+B(x+2)+C(1+2x)$

$x=-2,\qquad 5(-2)+4=C(1+2(-2))$

[M1 – Substitute with either $x=-2, x=-\frac{1}{2}$]

$-6=-3C$

$C=2$

$x=-\frac{1}{2},\qquad 5\left(-\frac{1}{2}\right)+4=A(2+\left(-\frac{1}{2}\right))^2$

$A=\frac{2}{3}$

[M1 – 2 out of 3 values of A,B, C correct]

$x=0,\qquad 4=\frac{2}{3}(4)+B(2)(1)+2(1)$

$B=-\frac{1}{3}$

$$\frac{5x+4}{(1+2x)(x+2)^2}=\frac{2}{3(1+2x)}-\frac{1}{3(x+2)}+\frac{2}{(x+2)^2}$$

[A1]

(b) **Hence**, find $\int\frac{5x+4}{(1+2x)(x+2)^2}\,dx$

[3]

$$\int \frac{2}{3(1+2x)} - \frac{2}{3(x+2)} + \frac{2}{(x+2)^2}\, dx$$

$$= \frac{1}{3}\int\left[\frac{2}{1+2x}\right]dx - \frac{1}{3}\ln(x+2) + \frac{2(x+2)^{-2+1}}{(-1)(1)} + c$$

$$= \frac{1}{3}\ln(1+2x) - \frac{1}{3}\ln(x+2) - \frac{2}{(x+2)} + c$$

[M1 cao] – $\frac{1}{3}\ln(1+2x)$ seen

[M1 cao] - $-\frac{1}{3}\ln(x+2)$ seen

[M1 cao] - $-\frac{2}{(x+2)}$ seen

8 Elvin bought a new car and the value, $\$V$, of the car is given by the formula, $V = 130000e^{-mt}$ where m is a constant and t is the age of the car in **months**.
The value of the car after 3 years is expected to be \$100 000.

(i) Find the amount that Elvin paid for the car? [1]

When $t = 0$,

$V = 130000e^{0}$

$V = \$130,000$

(ii) Calculate the value of the car after 65 months. Giving your answer to the nearest dollars. [3]

$100000 = 130000e^{-(36)m}$ [M1] Attempt to find value of m

$\frac{10}{13} = e^{-36m}$

$\ln\left(\frac{10}{13}\right) = -36m$

$m = 0.00729$ (3 sig. fig.) [M1 cao] Value of m found

$V = 130000e^{-(0.00729)(65)}$

$V = 80949.15 \approx \$80,949$ [A1]

(iii) In which month did the value of the car first depreciate to $30,000. Giving your answer to the nearest month. [3]

$30000 = 130000e^{-(0.00729)(t)}$ [M1]

$\frac{3}{13} = e^{-(0.00729)(t)}$

$\ln\left(\frac{3}{13}\right) = -0.00729t$

[M1] Taking ln on both sides

$t = 201.14\text{months}$

$t = 201$ month [A1]

9 The equation of a circle C is $x^2 + y^2 - 4x - 6y + 4 = 0$.

(i) Find the coordinates of its centre and the radius of the circle. [3]

$2g = -4$ $2f = -6$

$g = -2$ $f = -3$ [M1 To find g and f]

Centre $(2,3)$ [A1] Centre

Radius $= \sqrt{(-2)^2 + (-3)^2 - 4}$ = 3 units [A1] Radius

(ii) Explain whether the point $A(4,5)$ lies outside the circle, inside the circle or on the circle. [2]

Distance between Centre of circle to A = $= \sqrt{(2)^2+(2)^2} = 2.83$ units [M1]

Since the distance between the centre of circle to A (2.83 units) is less than the radius (3 units) of the circle, therefore the point A will lie <u>inside the circle.</u>

[M1] With correct explanation

(iii) Find the equation of circle D, which is a reflection of circle C along the y-axis. [2]

New centre of the circle $D = (-2, 3)$ [M1] New centre of circle D

Equation of Circle D : $(x+2)^2 + (y-3)^2 = 9$

OR

$x^2 + y^2 + 2x - 6y + 4 = 0$ [A1]

10

(a) Without using calculator, simplify $\dfrac{\sqrt{162}+\sqrt{72}}{\sqrt{54}}$. [2]

$$\frac{\sqrt{162}+\sqrt{72}}{\sqrt{54}}=\frac{\sqrt{81}\sqrt{2}+\sqrt{36}\sqrt{2}}{\sqrt{9}\sqrt{6}}$$

[M1] Break up indices

$$=\frac{9\sqrt{2}+6\sqrt{2}}{3\sqrt{2}\sqrt{3}}$$

$$=\frac{15\sqrt{2}}{3\sqrt{2}\sqrt{3}}$$

$$=\frac{5}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{5\sqrt{3}}{3}$$

[B1]

(b) The area of a rectangular plot of land is $(31+8\sqrt{2})$ cm^2.

The length of the rectangular land is $(3+\sqrt{2})^2$ cm.

Find the breadth of the rectangular land in the form $a+\sqrt{b}$, where a and b are integers. [4]

$$\frac{31+8\sqrt{2}}{\left(3+\sqrt{2}\right)^2}=\frac{31+8\sqrt{2}}{2+6\sqrt{2}+9}$$

[M1 cao] Area / Length

$$=\frac{31+8\sqrt{2}}{11+6\sqrt{2}}$$

$$=\frac{31+8\sqrt{2}}{11+6\sqrt{2}}\times\frac{11-6\sqrt{2}}{11-6\sqrt{2}}$$

[M1] Rationalise

$= \frac{341 - 186\sqrt{2} + 88\sqrt{2} - 96}{(11)^2 - (36)(2)}$ [M1] Correct Expansion

$= \frac{245 - 98\sqrt{2}}{49}$

$= 5 - 2\sqrt{2}$ [B1]

11 In a particular probiotic drink, it contains live and active "good bacteria".
The population y, in millions, is recorded in a laboratory experiment for 20 minutes.

It is given by $y = Ax^n + 8$ where x is the time measured in minutes, A and n are constants.
The results are shown in the table below.

x	4	8	12	16	20
y	292	1138	8118	17914	35735

(a) Using the graph grid on the following page,

Plot $\lg(y-8)$ against $\lg x$, draw a straight line graph. [3]

(b) Use your graph to estimate the

(i) correct value of y for which an error has been made. [2]

(ii) value of A and of n, [3]

12 (a) Show that $\cos^2 52.5° - \sin^2 52.5° = \cos 105°$.
[1]

$$\cos^2 52.5° - \sin^2 52.5° = \cos 2(52.5°)$$
$$= \cos 105°$$ [AG1]

(b) **Without using calculator**, find the value of $\cos^2 52.5° - \sin^2 52.5°$. [5]

$$\cos 105° = \cos(45° + 60°)$$

$$= \cos 45° \cos 60° - \sin 45° \sin 60°$$ [M1] Apply addition formulae

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)$$ [M1] Exact values for special angles

$$= \left(\frac{1}{2\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)$$

$$= \left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right)$$ [M1] $\left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right)$ seen

$$= \frac{1-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$ [M1] For Rationalisation

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$ [A1]

13 **(a)** Prove the identity $\sin^2 2x\left(\cot^2 x-\tan^2 x\right)=4\cos 2x$. [3]

$\text{LHS}=\sin^2 2x\left(\cot^2 x-\tan^2 x\right)$

$=(2\sin x\cos x)^2\left(\cot^2 x-\tan^2 x\right)$

[M1] Involve any trigo identities

$=(2\sin x\cos x)^2\left(\dfrac{\cos^2 x}{\sin^2 x}-\dfrac{\sin^2 x}{\cos^2 x}\right)$

[M1] Involve any trigo identities

$=(2\sin x\cos x)^2\left(\dfrac{\left(\cos^2 x\right)^2-(\sin^2 x)^2}{\left(\sin^2 x\right)(\cos^2 x)}\right)$

$=4\left[(\cos^2 x-\sin^2 x)(\cos^2 x+\sin^2 x)\right]$

$=4\left[(\cos^2 x-\sin^2 x)(1)\right]$

$=4\cos 2x$ [A1] Apply $\cos^2 x-\sin^2 x=\cos 2x$

$=\text{RHS}$ **(PROVED)**

(b) **Hence**, evaluate $\int_0^{\frac{\pi}{2}}\left[\sin^2 2x(\cot^2 x-\tan^2 x)+\tan^2 2x\right]\mathrm{d}x$ [4]

$\int_0^{\frac{\pi}{2}}\left[4\cos 2x+(\sec^2 2x-1)\right]\mathrm{d}x$

[M1] Use of $1+\tan^2 2x=\sec^2 2x$ or replace with 4 cos2x

$=\left[4\left(\frac{\sin 2x}{2}\right)+\frac{1}{2}(\tan 2x)-x\right]_0^{\frac{\pi}{2}}$

[M1] Integrate either $4\cos 2x$ or $\sec^2 2x$ correctly

$=\left[2\sin 2x+\frac{1}{2}(\tan 2x)-x\right]_0^{\frac{\pi}{2}}$

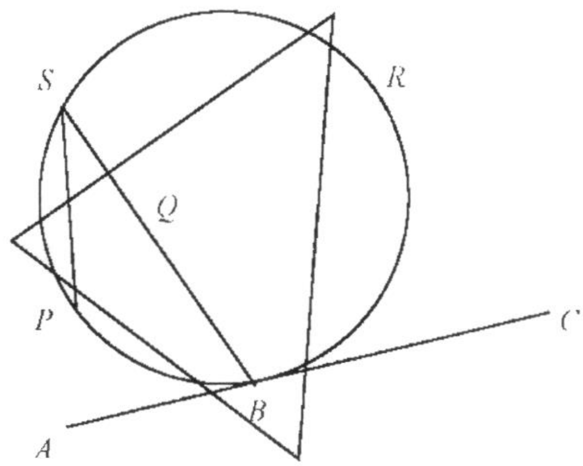
$=\left[(2\sin\pi-\frac{1}{2}\tan\pi-\frac{\pi}{2})-(2\sin 0-\frac{1}{2}\tan 0-0)\right]$ [M1] Concept of substituting limits

$=\left[(0-0-\frac{\pi}{2})-(0-0-0)\right]$

$=-\frac{\pi}{2}$

[A1]

14 In the diagram below, AC is a tangent to the circle at B. SB is a chord of the circle. Given that BP and BR bisects angle SBA and SBC respectively.



(i) Prove that $\angle PBR = 90^\circ$. [3]

Let $\angle ABP = x, \angle CBR = y$

Therefore, $\angle ABP = \angle PRB = x$ (Alternate segment theorem)

$\angle CBR = \angle BPR = y$ (Alternate segment theorem) [M1] Alt seg theorem

$\angle ABP = \angle PBS = x$ (BP bisects $\angle SBA$)

$\angle CBR = \angle RBS = y$ (BR bisects $\angle SBC$) [M1] Bisects angle

Consider Triangle PRB, $2x + 2y = 180^\circ$ (sum of angles in a triangle)

$x + y = 90^\circ$ [M1] $x + y = 90^\circ$

(ii) Explain why *PR* is the diameter of the circle. [1]

Therefore, $\angle PBR = 90°$,
Thus, *PRB* is a right angle triangle in a semi circle. [A1] Triangle in semi circle

Hence, *PR* is the diameter of the circle.

(iii) Show that ΔSPQ is similar to ΔRBQ.
[2]

$\angle BRP = \angle BSP$ (alternate segment theorem) [M1- Any 1 reason given]

$\angle SQP = \angle RQB$ (vertically opposite angle)

By *AA*, ΔSPQ is similar to ΔRBQ. [AG1 – State by *AA*]

It is given that $4PQ = QR$ and $SQ = QB$.

(iv) Prove that $\left(\frac{QB}{PQ}\right)^2 = 4$. [2]

Since ΔSPQ is similar to ΔRBQ ,

$$\frac{SQ}{RQ}=\frac{PQ}{BQ}$$

$$SQ\times QB=PQ\times QR \qquad \text{[M1]}$$

$$(QB)\times QB=PQ\times(4PQ)$$

$$(QB)^2=4PQ^2$$

$$\left(\frac{QB}{PQ}\right)^2=4$$

Pasir Ris Secondary School

Name	Class	Register Number
MARK SCHEME		

SECONDARY 4 EXPRESS / 5 NORMAL ACADEMIC

PRELIMINARY EXAMINATION 2023

ADDITIONAL MATHEMATICS 4049/02

Paper 2 **25 August 2023**
Friday 0800-1015 2 hours 15 minutes

Additional Materials: Electronic calculator

READ THESE INSTRUCTIONS FIRST

Write your name, class and register number on all the work you hand in.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

If working is needed for any question, it must be shown with the answer. Omission of essential working will result in loss of marks.

You are expected to use a scientific calculator to evaluate explicit numerical expressions.
If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures. Give answers in degrees to one decimal place.

For π , use either your calculator value or 3.142, unless the question requires the answer in terms of π .

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is **90**.

This document consists of **24** printed pages.

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx = c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\text{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

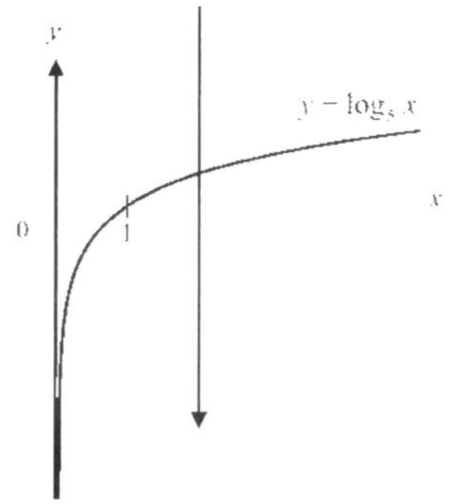
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}ab\sin C$$

Answer **all** the questions.

1 **(a)** Sketch the graph of $y = \log_5 x$, indicating the coordinates of the point where the curve $y = \log_5 x$ cuts the x-axis. [2]



B1 for correct shape

B1 for x-value of 1 shown on graph

(b) Given that $\lg(7y+2) - x^2 = 2$, express y in terms of x. [3]

$\lg(7y+2) - x^2 = 2$

$\lg(7y+2) = 2 + x^2$ M1CAO

$7y + 2 = 10^{2+x^2}$ M1 to change lg to base 10

$7y = 10^{2+x^2} - 2$

$\therefore y = \dfrac{10^{2+x^2} - 2}{7}$ A1

(c) Solve the equation $\log_3(2x^2 + 27) - 2\log_9(x - 12) = 1$. [4]

$$\log_3(2x^2+27)-2\log_9(x-12) = 1$$

$$\log_3(2x^2+27)-2\left(\frac{\log_3(x-12)}{\log_3 9}\right) = 1$$ M1 for change of base

$$\log_3(2x^2+27)-2\left(\frac{\log_3(x-12)}{\log_3 3^2}\right) = 1$$

$$\log_3(2x^2+27)-2\left(\frac{\log_3(x-12)}{2(1)}\right) = 1$$ M1 for $\log_3 3=1$ or combine two terms using log rule

$$\log_3(2x^2+27)-\log_3(x+12) = 1$$

$$\log_3\frac{(2x^2+27)}{(x+12)} = 1$$

$$\frac{(2x^2+27)}{(x+12)} = 3$$ M1CAO

$$2x^2+27 = 3(x+12)$$

$$2x^2-3x-9 = 0$$

$$(2x+3)(x-3) = 0$$

$$\therefore x = -1.5 \quad \text{or} \quad x = 3$$ A1

2 The expression $3x^3 + ax^2 + bx + 2$, where a and b are constants, has a factor of $x - 1$. When the expression is divided by $x - 2$, it leaves a remainder that is 2.5 times the remainder when divided by $x + 1$.

(a) Show that $a = 2$ and $b = -7$. [5]

$f(1) = 3x^3 + ax^2 + bx + 2$
$3(1)^3 + a(1)^2 + b + 2 = 0$
$a + b = -5 \quad -(1)$ M1 or showing f(1) = 0

$f(2)$:
$3(2)^3 + a(2)^2 + b(2) + 2 = 4a + 2b + 26$

M1CAO for either *f*(2) or *f*(-1)

$f(-1)$:
$3(-1)^3 + a(-1)^2 + b(-1) + 2 = a - b - 1$

$f(2) = 2.5f(-1)$
$4a + 2b + 26 = 2.5(a - b - 1)$ M1 for equating
$1.5a + 4.5b = -28.5 \quad -(2)$

From (1), $a = -5 - b \quad -(3)$

sub (3) into (2)
$1.5(-5 - b) + 4.5b = -28.5$
$-7.5 - 1.5b + 4.5b = -28.5$
$3b = -21$
$b = -7$ A1

Sub $b = -7$ into (3)
$a = -5 - (-7) = 2$ A1

$\therefore a = 2, b = -7$ shown

(b) Hence, **by using long division**, solve the equation $3x^3 + ax^2 + bx + 2 = 0$. [3]

$$
\begin{array}{r}
3x^2 + 5x - 2 \\
x-1\overline{)3x^3 + 2x^2 - 7x + 2} \\
-)\,3x^3 - 3x^2 \qquad\qquad \\
5x^2 - 7x + 2 \\
-)\,5x^2 - 5x \qquad \\
-2x + 2 \\
-)\,-2x + 2 \\
0
\end{array}
$$

M1 for showing at least 1 level correct long division

$3x^3 + 2x^2 - 7x + 2 = 0$

$(x-1)(3x^2 + 5x - 2) = 0$

$(x-1)(3x-1)(x+2) = 0$ M1 for factorising

$\therefore x = 1 \quad x = \frac{1}{3} \quad x = -2$ A1

3 The equation of a curve is $y = 1 - 3\cos 2x$ for $0 < x < 2\pi$.

(a) State the period of f. [1]

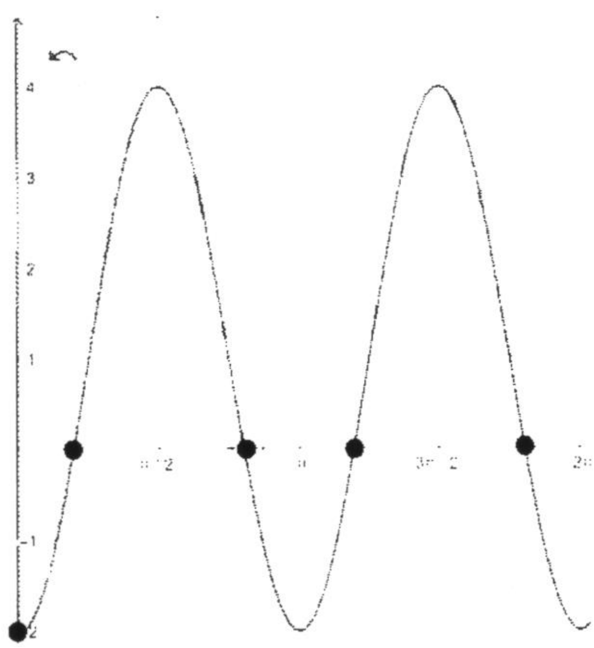
Period $\frac{2\pi}{2} = \pi$ B1, award if answer in degrees given also. i.e. 180°

(b) State the maximum and minimum values of y. [2]

Max value = 4 B1

Min value = -2 B1

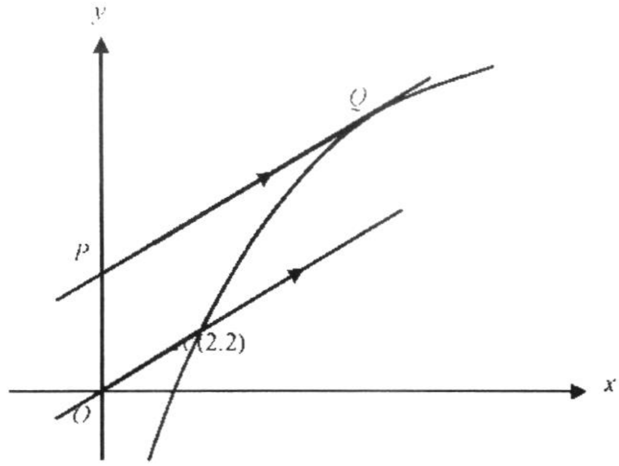
(c) Sketch the graph of $y = 1 - 3\cos 2x$ for $0 \le x \le 2\pi$, indicating the number of points of intersections with the axes clearly. [2]



B1 for correct max, min values and shape

B1 for correct 5 points of intersection of curve with axes

4



The diagram shows part of the curve and two parallel lines OR and PQ.

The line OR intersects the curve at the point $R(2,2)$ and the line PQ is a tangent to the curve at the point Q.

(a) Show that the coordinates of P and Q are $(0,4)$ and $(4,8)$ respectively. [4]

$$\frac{y-8}{0-4}=1$$

$$y-8=-4$$

$$y=4$$

M1 for correct gradient

$m_{PQ}=m_{OR}=1$

$\therefore$ Coordinates of $P(0,4)$ A1

$y = 10 - \frac{32}{x^2}$

$\frac{dy}{dx} = 64x^{-3}$ M1CAO for correct differentiation

$64x^{-3} = 1$

$\frac{64}{x^3} = 1$

$x^3 = 64$

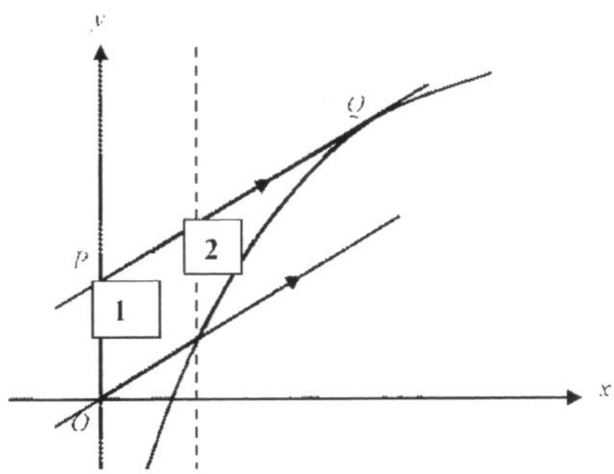
$x = 4$

Sub $x = 4$ into y

$y = 10 - \frac{32}{4^2} = 8$

$\therefore$ Coordinates of $Q(4,8)$ M1CAO

(b) Find the area of the shaded region $OPQR$. [6]



To find shaded area 1,

Area of trapezium − Area of triangle

$= \frac{1}{2}(2)(4+6) - \frac{1}{2}(2)(2)$ M1

$= 8 \text{units}^2$ M1

To find shaded area 2,

Area of trapezium − Area under curve from $x = 2$ to $x = 4$

$= \frac{1}{2}(2)(6+8) - \int_2^4 10 - \frac{32}{x^2}\, dx$ M1

$= 14 - \left[10x + \frac{-32x^{-1}}{-1}\right]_2^4$ M1CAO for correct integration

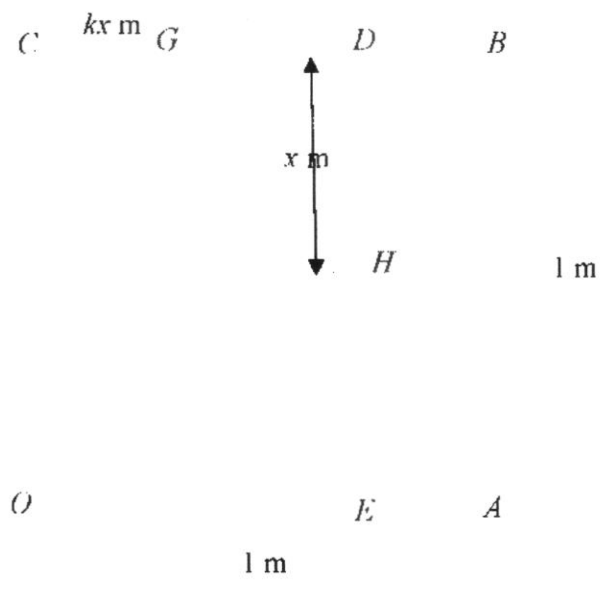
$= 14 - \left[\left(40 + \frac{32}{4}\right) - \left(20 + \frac{32}{2}\right)\right]$

$= 14 - 12$

$= 2\text{units}^2$ M1

Total area = 8 + 2 = 10units^2 A1

5



In the diagram, $OABC$ is a square of side 1m. D and E are points on BC and AO respectively. DE is parallel to the sides of CO and BA. H is a point on DE and G is a point on CB such that DH is x m and CG is kx m, where k is a constant and $0 < k < 1$.

(a) Show that the sum, S m^2, of areas of triangles OHG and ABH, is given by

$$S = \frac{1}{2}\left(1 - kx + kx^2\right)$$

[3]

Area of ΔOHG and ΔABH

$= 1 - (\text{Area of } \Delta OCG + \text{Area of } \Delta BGH + \text{Area of } \Delta AOH)$

$= 1 - \left(\frac{1}{2} \times 1 \times kx + \frac{1}{2} \times x \times (1 - kx) + \frac{1}{2}(1 - x)(1)\right)$ M1CAO for area of square = 1m^2

M1 for area of any 1 triangle found

$= \frac{1}{2} - \frac{1}{2}kx + \frac{1}{2}kx^2$ A1

$= \frac{1}{2}(1 - kx + kx^2)$ shown A.G.

(b) Find the value of x such that S a minimum. [3]

$\frac{dS}{dx} = -\frac{1}{2}k + kx$ M1CAO for

$\frac{dS}{dx} = 0$ M1CAO for

$-\frac{1}{2}k + kx = 0$

$x = \frac{1}{2}$

$\frac{d^2S}{dx^2} = k > 0$ (minimum) A1

(c) Given that x can vary, find in terms of k, the minimum value of S. [1]

$S=\frac{1}{2}\left(1-\frac{1}{4}k\right)$ B1

6 (a) It is given that $\sin\theta=\frac{1}{\sqrt{1+a^2}}$ and θ is acute. Obtain an expression, in terms of a, for $2\tan\theta$. [2]

$\tan\theta=\frac{1}{a}$ M1

$\therefore 2\tan\theta$

$=\frac{2}{a}$ A1

(b) Solve the equation $2\sec A=\tan^2 A-2$ for $0° < A < 360°$ [4]

$2\sec A = \tan^2 A - 2$

$2\sec A = \sec^2 A - 1 - 2$ M1CAO for using

$\sec^2 A - 2\sec A - 3 = 0$

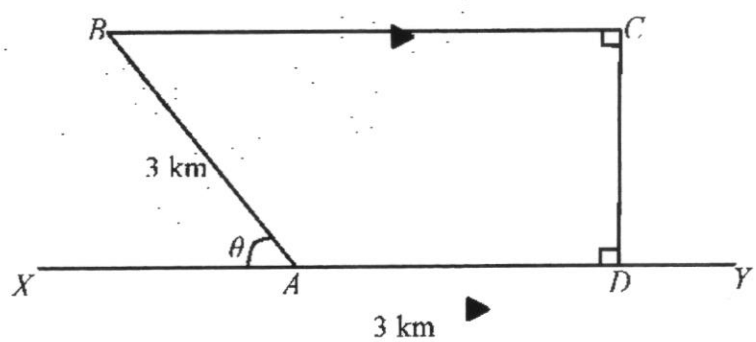
$(\sec A - 3)(\sec A + 1) = 0$ M1 for factorising

$\cos A = \frac{1}{3}$ or $\cos A = -1$ M1

$\alpha = 70.5°$ $\alpha = 0°$

$\therefore A = 70.5°, 180°, 289.4°$ A1

7 Charles signed up for a race and was given a brochure showing the race route.



XY is a straight road. Participants would start running from point A to D, then from D to C, followed by C to B and finally from B back to A. BC is parallel to XY. CD is perpendicular to both BC and XY. $AB = AD = 3$ km and angle XAB is θ °. The total distance of the route is L km.

(a) Show that L can be expressed as $p\cos\theta + q\sin\theta + r$, where p, q and r are constants. [2]

$CD = 3\sin\theta$
$BC = 3\cos\theta + 3$ M1CAO for both

$\therefore L = 3\cos\theta + 3\sin\theta + 3 + 3 + 3$
$= 3\cos\theta + 3\sin\theta + 9$ A1

(b) Express L in the form $R\cos(\theta - \alpha) + r$, where $R > 0$ and α is an acute angle. [3]

$L = 3\cos\theta + 3\sin\theta + 9$
$= R\cos(\theta - \alpha) + 9$

$R = \sqrt{3^2 + 3^2} = \sqrt{18}$ M1
$\alpha = \tan^{-1}\frac{3}{3} = 45°$ M1

$\therefore L = \sqrt{18}\cos(\theta - 45°) + 9$ A1

(c) The total length of the route is found to be 13 km. Find the possible values of θ. [4]

$\sqrt{18}\cos(\theta - 45°) + 9 = 13$ M1CAO
$\sqrt{18}\cos(\theta - 45°) = 4$
$\cos(\theta - 45°) = \frac{4}{\sqrt{18}}$
basic angle $= \cos^{-1}\frac{4}{\sqrt{18}} = 19.5°$ M1 for basic angle

$\theta - 45° = 19.5°, -19.5°$ M1 for correct quadrants
$\theta = 64.5°, 25.5°$
$\therefore \theta = 25.5°, 64.5°$ A1

(d) Charles claims that he can finish the race in under 49 minutes if he maintains his speed of 16 km/h throughout the race regardless of the value of θ. Is Charles's claim true? Explain your answer. [2]

Distance ran by Charles $= \frac{49}{60} \times 16 = 13.1$ km

M1 for both distance and max L

Maximum L occurs when $\cos(\theta - 45°) = 1$

Therefore, $L = \sqrt{18}(1) + 9 = 13.24$ km

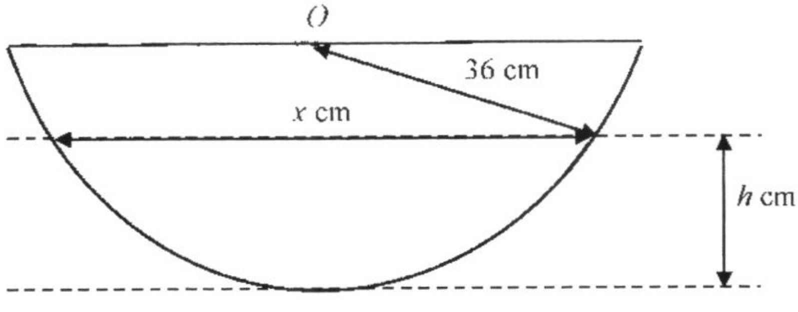
Since Charles covered a lesser distance than the maximum, therefore his claim is **not** true. A1

OR

Maximum time taken $= \frac{\sqrt{18}+9}{16} \times 60 = 49.6$ min M1

Since Charles's time exceeded 49 mins, therefore his claim is **not** true. A1

8



The diagram shows the cross section of a hemispherical container with centre O. The container is filled with water and has a radius of 36 cm. Water is leaked through a hole at the bottom of the container such that when the water in the container is at a depth of h cm, the horizontal water surface has a length of x cm.

(a) Show that $x = \sqrt{288h - 4h^2}$. [2]

Using Pythagoras theorem,

$$36^2 = (36-h)^2 + \left(\frac{x}{2}\right)^2 \qquad \text{M1}$$

$$36^2 = 36^2 - 72h + h^2 + \frac{x^2}{4}$$

$$x^2 = 288h - 4h^2 \qquad \text{A1}$$

$$x = \sqrt{288h - 4h^2} \quad \text{shown} \qquad \text{A.G}$$

(b) Given that the length of the horizontal water surface is decreasing at a rate of 4 cm/min, find the rate at which the depth of the water is decreasing when $h = 10$ cm. [5]

$\frac{dx}{dt} = -4$ cm/min M1CAO for decreasing rate

$\frac{dx}{dh} = \frac{1}{2}\left(288h - 4h^2\right)^{-\frac{1}{2}}(288 - 8h)$ M1 for differentiation

At $h = 10$ cm, M1 for sub of h 10 cm

$-4 = \frac{1}{2}\left(288(10) - 4(10^2)\right)^{-\frac{1}{2}}(288 - 8(10)) \times \frac{dh}{dt}$ M1 for

$\frac{dh}{dt} = -1.92$cm/min

$\therefore$ Depth is decreasing at 1.92 cm/min A1

9 A particle P moves along a horizontal straight line so that its displacement, s m, from a fixed point O, t seconds after motion has begun is given by $s = 14 + 5t - t^2 - t^3$.

(a) Find an expression, in terms of t, the velocity of P. [1]

$v = \frac{ds}{dt} = 5 - 2t - 3t^2$ B1

(b) Find the acceleration of P when it is instantaneously at rest. [4]

$v = 0$ M1CAO

$5 - 2t - 3t^2 = 0$

$(1-t)(3t+5) = 0$

$t = 1$ or $t = -\frac{5}{3}$ rej M1

$a = \frac{dv}{dt} = -2 - 6t$ M1CAO for differentiation of v

when $t = 1$,

$a = -2 - 6(1) = -8\text{ m/s}^2$ A1

A second particle Q moves along the same horizontal straight line as P and starts from O at the same instant when P begins to move. The initial velocity of Q is 6 m/s and its acceleration, a m/s^2, t seconds after motion has begun, is given by $a = 4 - 6t$.

(c) Find the value of t at the instant when P and Q collide. [5]

For particle Q:

$v = \int 4 - 6t \, dt$ M1CAO for integrating v

$= 4t - \frac{6t^2}{2} + c$

$= 4t - 3t^2 + c$

Initially, $t = 0, v = 6\text{m/s}, s = 0\text{m}$

$6 = 4(0) - 3(0)^2 + c$

$c = 6$

$\therefore v = 4t - 3t^2 + 6$ M1

$s = \int 4t - 3t^2 + 6 \, dt$

$= \frac{4t^2}{2} - \frac{3t^3}{3} + 6t + c$

$= 2t^2 - t^3 + 6t + c$

$0 = 2(0)^2 - (0)^3 + 6(0) + c$

$c = 0$

$\therefore s = 2t^2 - t^3 + 6t$ M1

When particles P and Q collide, displacement is equal

$14 + 5t - t^2 - t^3 = 2t^2 - t^3 + 6t$ M1 for equating the 2 displacements

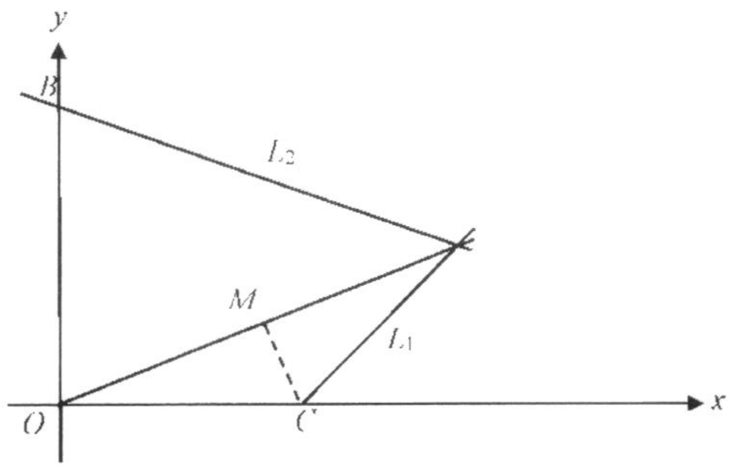
$3t^2 + t - 14 = 0$

$t = 2$ or $t = -\frac{7}{3}$ rej A1

(d) Determine whether P and Q are travelling in the same direction at this instant. [1]

They are travelling in the opposite direction at this instant when they collide. B1

10



The diagram shows lines L_1 and L_2 intersecting at $A(16, k)$. Line L_1 has equation $3y = 4x - 40$. The point M is the midpoint of OA. The line L_2 has a y-intercept of 48 and the line L_1 intersects the x-axis at C.

(a) Show that angle OMC is $90°$. [5]

Sub $(16, k)$ into $3y = 4x - 40$

$3k = 4(16) - 40$

$k = 8$

$\therefore$ coordinates of $A(16, 8)$ M1CAO

Sub $(x, 0)$ into $3y = 4x - 40$ (cuts x axis)

$3(0) = 4x - 40$

$x = 10$

$\therefore$ coordinates of $C(10, 0)$ M1CAO

Using mid-point formula.

$M\left(\frac{0+16}{2}, \frac{0+8}{2}\right) = (8, 4)$ M1

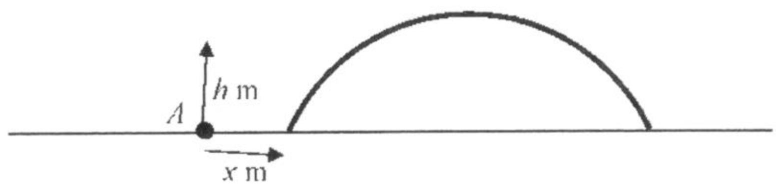
$m_{OA} = \frac{8-0}{16-0} = \frac{1}{2}$

M1 for either or

$m_{MC} = \frac{4-0}{8-10} = -2$

$\therefore$ Since $m_{OA}m_{MC} = -1$, angle OMC is $90°$. shown A1 with

11 A skateboard park has a mould that is h metres high at the point where the horizontal distance, from a fixed point A, is x metres.



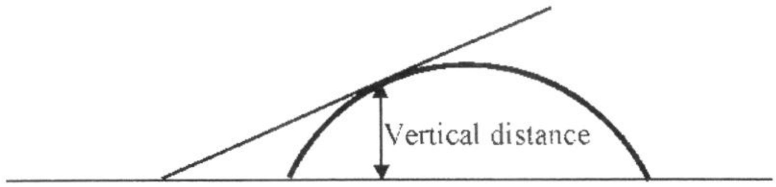
The height of the mould can be modelled by $h = -x^2 + 6x - 3$ for $1 < x < 5$.

(a) By **only completing the square**, find the height of the mould at its **highest** point. [2]

$$h = -x^2 + 6x - 3$$
$$= -(x^2 - 6x + 3)$$
$$= -\left[\left(x - \frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 3\right]$$ M1 for completing the square
$$= -(x-3)^2 + 9 - 3$$
$$= -(x-3)^2 + 6$$

$\therefore$ The height of the mould at its highest point is 6 m A1

(b) A ramp built up the side of the mould is a tangent to the mould.



The ramp can be modelled by the function $h = 2.5x + c$.

Find the vertical distance where the ramp will meet the mould. (*Ignore the thickness of the ramp.*) [3]

Gradient = 2.5

$\frac{dh}{dx} = -2x + 6$ M1CAO

$-2x + 6 = 2.5$

$x = \frac{7}{4}$ M1

$\therefore h = -(\frac{7}{4})^2 + 6(\frac{7}{4}) - 3$

$= 4.4375$ m A1

(c) The height h metres of a skateboard path at a horizontal distance r metres from another point B, can be modelled by the function $h=\frac{2}{3}r^3-4r^2+6r$, $0<r<4$. For the safety of the skaters, there is a height regulation that requires no part of the skateboard path to be 3 metres above the ground.

Fully describe this curve, including its **turning points**, the **horizontal distance r metres from point B**, and state whether the skateboard path complies with the height regulations. [3]

$h=\frac{2}{3}r^3-4r^2+6r$

At stationary point, $\frac{dh}{dr}=0$

$2r^2-8r+6=0$ M1CAO for expression

$2(r-1)(r-3)=0$

$r=1 \text{ or } 3$

$r=1;\ h=\frac{8}{3}$

$r=3;\ h=0$

$\frac{d^2h}{dr^2}=4r-8$

at $r=1;\ h=\frac{8}{3} \rightarrow \frac{d^2h}{dr^2}<0 \Rightarrow$ maximum point M1 for 1st or 2nd derivative test for both points

at $r=3;\ h=0 \rightarrow \frac{d^2h}{dr^2}>0 \Rightarrow$ minimum point

at $r=4;\ h=\frac{8}{3}$

The curve has a maximum stationary point of $h=\frac{8}{3}$ m at $r=1$ and a minimum stationary point of $h=0$ m at $r=3$ and a height of $\frac{8}{3}m$ at $r=4$.

$\therefore$ The skateboard path complies with the height regulation. A1